Congruence and Conjunctivity of Matrices to Their Adjoints

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ABSTRACT

Every square matrix over a field F is involutorily congruent over F to its transpose, and hence each such matrix is the product of a symmetric matrix and an involutory matrix over F. In the usual complex case every matrix which is conjunctive with its adjoint (=conjugate-transpose) is involutorily conjunctive with its adjoint and hence is the product of a hermitian matrix and an involutory matrix; furthermore every such matrix is conjunctive with a real matrix. These three conditions on a matrix, (1) being conjunctive with its adjoint, (2) being involutorily conjunctive with its adjoint, and (3) being conjunctive with a real matrix, are studied in the more general context of a field F with involution, and it is shown in general that (3) implies (2), that (2) implies (3) if char $F \neq 2$ (a 2×2 counterexample exists for each F with char $F=2$), and that (1) does not in general imply (2) (a 2×2 counterexample in the complexification of the rational field is presented). The problem of deciding which matrices satisfy (2) is equivalent (even in this general context) to the problem of deciding which pairs of self-adjoint ("hermitian") matrices are involutorily conjunctive. For the general 2×2 case, the three conditions are characterized in terms of norms.

1. INTRODUCTION AND REDUCTION TO THE NONSINGULAR CASE

In this paper we study three properties which a square matrix may or may not have. We wish to treat both the case of an arbitrary field and the case of an arbitrary "field with involution." To do this conveniently, we adopt the

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terminology in [1], which we briefly recapitulate here. A pair (F, E) of fields F and E is *complic* provided F is a proper separable quadratic extension of E, (F, E) is *simplic* provided $F = E$, and (F, E) is *admissible* provided it is complic or simplic. Associated with each admissible (F, E) is an involutory F-automorphism $\alpha \mapsto \bar{\alpha}$ (called (F, E)-*conjugation*) whose fixed field is E. As usual, the entrywise conjugate of a matrix S is denoted by $\bar{S}$, the transpose of S is denoted by S', and $\bar{S}'$ is denoted by S^*.

Matrices S and T are (F, E)-*congruent* (or $*$-*congruent* when the pair (F, E) is understood from context) provided $C^*SC = T$ for some nonsingular C over F. Thus $*$-congruence is conjunctivity in the usual complex case and is ordinary congruence in the simplic cases. When dealing specifically with the simplic cases, we shall usually write S^* as S' and "$*$-congruence" as "congruence," but otherwise we shall retain the $*$ in the notation.

The three matrix concepts that are the subject of this paper are next defined in their general setting:

DEFINITION 1.1. Let (F, E) be an admissible pair and S be a square matrix over F. Then S is (F, E)-*conjoint* provided S and S^* are $*$-congruent, S is (F, E)-*injoint* provided S and S^* are involutorily $*$-congruent (i.e., provided $C^*SC = S^*$ for some $C = C^{-1}$ over F), and S is (F, E)-*rejunctive* provided S is $*$-congruent to a matrix over E.

Thus every $*$-symmetric matrix (that is, every matrix $S = S^*$) over F is (F, E)-injoint and (F, E)-rejunctive. Other examples and counterexamples occur in Sections 3 and 5.C.

We adopt the following conventions: Throughout this paper (F, E) will be an admissible pair (arbitrary except as specified), and except as otherwise specified, all scalars will be in F (specific exceptions: we shall continually deal with scalar polynomials in $F[t]$ with t an indeterminate over E, and in Section 5.C we shall deal with a scalar field K which is quadratic over F), all matrices and polynomials will be over F (exceptions: we shall sometimes refer to matrix pencils $S + tT$ and, in Section 5.C, to matrices over K), and, except as specified in Notation 1.10(a) and Section 5.C, all conjugates will be (F, E)-conjugates, and likewise for conjointness, injointness, and rejunctivity. Occasionally we shall write "in F" or "over F" or "(F, E)" even where the above conventions make these expressions redundant.

Obviously every injoint matrix is conjoint, and in the simplic case every square matrix is rejunctive. The nomenclature was motivated by the usual complex case, where S conjoint means S is *con*junctive with its ad*joint* S^*, S injoint means S is *in*volutorily con*joint*, and S rejunctive means there is a *re*al matrix con*junctive* with S.

The study of conjointness and rejunctivity was motivated by [5, Section 2], which reports on an analogous study for similarity (i.e., similarity of A and

A^*, and similarity of a matrix over F to a matrix over E), and the inclusion of injointness in this study was motivated by [6], which extended the analogy to the factorization ($A=HK$ with H and K $*$-symmetric) occurring in [5] (see Fact 1.4 and the paragraph preceding it).

Next we give an alternate characterization for rejunctivity which will be used frequently in this paper. In this characterization an important role is played by matrices $C=\bar{C}^{-1}$. Such matrices are called *pseudo-involutory* in [2, Remark 2.5] and *circular* (at least in the usual complex case) in [15, p. 75]; we shall use the former term. (Of course pseudo-involutory = involutory in the simplic cases.)

FACT 1.2. *A matrix S is rejunctive iff $C^*SC=\bar{S}$ for some $C=\bar{C}^{-1}$ (i.e., iff S is pseudo-involutorily $*$-congruent to its conjugate).*

Proof. Through use of [9, Theorem 2′] or [2, Theorem 2.3], the proof in the complic case becomes a routine exercise in formal manipulation.

REMARK 1.3. The "formal manipulation" suggested for the proof of Fact 1.2, together with [2, Theorem 2.3], makes it clear that S is rejunctive iff C^*SC is over E for some nonsingular $C=C'$, i.e., iff S is symmetrically $*$-congruent to a matrix over E.

Formal manipulation of the definition of injointness leads to an interesting factorization for all injoint matrices, which is "dual" (in a sense proposed by Choi [6]) to factorizations for (1) a matrix A similar to A^* [5, Corollary], and (2) a matrix similar to its inverse [19, Theorem 1; 8]. (Coincidentally, the results for (1) and (2) are used latter in this paper.) We state the injointness result in several slight variations:

FACT 1.4.

(a) *For a square matrix S the following three statements are equivalent:* (i) *S is injoint;* (ii) *$SV=(SV)^*$ for some $V=V^{-1}$; and* (iii) *$S=HV$ for some $H=H^*$ and some $V=V^{-1}$.*

(b) *For a nonsingular matrix S each of* (i), (ii), (iii) *is equivalent to each of the following two statements:* (iv) *$SH^{-1}=(SH^{-1})^{-1}$ for some $H=H^*$; and* (v) *$S=HS^{-1}H$ for some $H=H^*$ (in words: S is $*$-symmetrically $*$-congruent to its inverse).*

Unsurprisingly, formal manipulation of the conjointness definition yields a formally weaker factorization:

FACT 1.5. *A square matrix* S *is conjoint iff* $S=TC$ *with* $TC=T^*C^{-1}$. *A nonsingular matrix* S *is conjoint iff* $S=TC$ *with* $T^{*-1}T$ (*the* cosquare *of* T [6, *p*. 417; 2]) *equal to* C^{-2} (*the inverse square of* C), *iff* S *is* $*$*-congruent to* S^{-1}.

REMARK 1.6. Another formal manipulation, which however applies only in the complic case, shows that the injointness concept is essentially equivalent to the concept of involutory $*$-congruence of two $*$-symmetric matrices, and that a corresponding equivalence holds for conjointness. Namely, let (F, E) be complic, $\alpha^2 \neq \bar{\alpha}^2$ (there always exists such an $\alpha \in F$), and $H=H^*$ and $K=K^*$. Then the following two statements are equivalent to each other:

(1) there is a nonsingular C such that $C^*HC=K$ and $C^*KC=H$,
(2) $\alpha H+\bar{\alpha}K$ is conjoint.

Also the following two statements are equivalent to each other:

(3) there is a $C=C^{-1}$ such that $C^*HC=K$,
(4) $\alpha H+\bar{\alpha}K$ is injoint.

The proofs are formal manipulations; for the proofs of (2)$\Rightarrow$(1) and (4)$\Rightarrow$(3) it helps to put $S=\alpha H+\bar{\alpha}K$ and solve formally for H and K in terms of S, S^*, and α.

Next we state some further elementary facts that we shall use frequently throughout the paper and that apply to every admissible (F, E).

FACT 1.7.

(a) (i) *Let* S *be* $n\times n$ *conjoint*, $\alpha \in E$, *and* B *be* $n\times n$ *nonsingular. Then* αS *and* B^*SB *are conjoint and* $\det S^2$ *is in* E. (ii) *Every direct sum* [15, *p*. 5] *of conjoint matrices is conjoint.*
(b) *Likewise with "conjoint" replaced by "injoint."*
(c) *Likewise with "conjoint" replaced by "rejunctive."*

Proof. We omit details, but one relevant calculation of great importance is the following: if C and B are nonsingular $n\times n$ and $B^*SB=T$, then

$$C^*SC=S^* \quad \text{implies} \quad (B^{-1}CB)^*T(B^{-1}CB)=T^*,$$

and

$$C^*SC=\bar{S} \quad \text{implies} \quad (B^{-1}C\bar{B})^*T(B^{-1}C\bar{B})=\bar{T}. \qquad \blacksquare$$

REMARK 1.8. Fact 1.7 tells us that each of the three properties we are studying (conjointness, injointness, and rejunctivity) is invariant under $*$-congruence. This implies that, in deriving necessary or sufficient conditions for a given matrix S to have a given one of these properties, we may replace S by any desired matrix $*$-congruent to S. Moreover, if S is nonsingular here, the familiar identity

$$B^{-1}(S^{*-1}S)B=(B^*SB)^{*-1}(B^*SB)$$

allows us, alternatively, to make this replacement so that the *cosquare* $S^{*-1}S$ of S is replaced by any desired matrix *similar* to $S^{*-1}S$. In general of course we cannot make *both* desired replacements simultaneously without further justification.

Our first application of Fact 1.7 is to obtain a partitioned form under $*$-congruence for injoint matrices when char $F\neq 2$.

FACT 1.9. If char $F\neq 2$, then S is injoint iff S is $*$-congruent to a matrix of the form

$$\begin{bmatrix} K & M^* \\ -M & L \end{bmatrix} \qquad \textit{with} \quad K=K^* \textit{ and } L=L^*$$

(but K and L may be of different sizes).

Proof. This is a routine exercise in partitioned matrices, using Facts 1.4(a) and 1.7(b), plus the well-known fact that, since char $F\neq 2$, every matrix $V=V^{-1}$ is similar to a direct sum $I\oplus -I$ for identity matrices of suitable sizes (which turn out to be suitable sizes for K and L as well). ∎

Next we collect for convenient reference some more or less standard terminology about polynomials, together with some elementary facts about them, which will be used in this paper.

NOTATION 1.10. Let t be an indeterminate over E, and $p(t)$ be in $F[t]$.

(a) Then $\bar{p}(t)$ is the polynomial in $F[t]$ defined by $\bar{p}(t)=\overline{p(t)}$, computed formally, using $\bar{t}=t$.

(b) If $p(t)$ has degree d and $p(0)\neq 0$, then $\tilde{p}(t)$ and $\hat{p}(t)$ are the (monic) polynomials in $F[t]$ defined (formally) by

$$\tilde{p}(t)=p(0)^{-1}t^d p(t^{-1}), \qquad \hat{p}(t)=\bar{p}(0)^{-1}t^d\bar{p}(t^{-1}).$$

REMARK 1.11. (a) The three mappings $f(t)\mapsto\bar{f}(t)$, $f(t)\mapsto\tilde{f}(t)$, and $f(t)\mapsto\hat{f}(t)$ are involutory automorphisms of the multiplicative monoid of monic polynomials of $F[t]$ with nonzero constant term. These mappings commute, and each is the composite of the other two. (b) The mapping $f(t)\mapsto\bar{f}(t)$ maps the system of elementary divisors and the system of invariant factors of a square matrix A onto the respective systems for $\bar{A}$ (and hence onto the respective systems for A^*). If A is nonsingular, then corresponding assertions apply to $f(t)\mapsto\tilde{f}(t)$ and $A\rightarrow A^{-1}$ (and $A\rightarrow A'^{-1}$), and to $f(t)\mapsto\hat{f}(t)$ and $A\rightarrow\bar{A}^{-1}$ (and $A\rightarrow A^{*-1}$). (c) A polynomial $p(t)$ in $F[t]$ is in $E[t]$ iff $p(t)=\bar{p}(t)$.

DEFINITION 1.12. A polynomial $p(t)$ in $F[t]$ is called *self-reciprocal* [19] provided $p(t)=\tilde{p}(t)$; it is called $*$*-self-reciprocal* [1] provided $p(t)=\hat{p}(t)$. A self-reciprocal polynomial of positive degree in $E[t]$ will be called $E[t]$-*irreducibly self-reciprocal* provided it is not the product of two self-reciprocal polynomials of positive degree in $E[t]$.

REMARK 1.13. (a) An $E[t]$-irreducibly self-reciprocal polynomial in $E[t]$ may well be $F[t]$-reducibly self-reciprocal, i.e., the product of two self-reciprocal polynomials of positive degree in $F[t]$, if (F,E) is complic. (b) Every self-reciprocal polynomial is monic, and its "constant term" is ± 1. (c) The only odd-degree irreducibly self-reciprocal polynomials are $t-1$ and $t+1$. (d) A polynomial $q(t)$ in $F[t]$ is $E[t]$-irreducibly self-reciprocal (in $E[t]$) iff there is a monic $F[t]$-irreducible $f(t)$ in $F[t]$ for which $q(t)$ is the lcm (least common multiple) of $\{f(t),\bar{f}(t),\tilde{f}(t),\hat{f}(t)\}$. (e) [19] Each self-reciprocal $p(t)$ in $E[t]$ has a factorization (unique up to order)

$$p(t)=q_1(t)^{m(1)}q_2(t)^{m(2)}\cdots q_k(t)^{m(k)},$$

where the k polynomials $q_1(t),\dots,q_k(t)$ are distinct $E[t]$-irreducibly self-reciprocal and the exponents $m(1),\dots,m(k)$ are positive.

Other notation and terminology used in this paper are as in [1] (which see for further details):

NOTATION 1.14.

(a) We denote by I_m the $m\times m$ identity matrix, by J_m the $m\times m$ lower-triangular nilpotent Jordan block, by G_m the $m\times m$ matrix whose rows are those of I_m but in reverse order. We often write I for I_m, J for J_m, or G for G_m when m is understood from context. Note that $G=G^*=G^{-1}$, $GJ=J^*G$, etc.

(b) We use $\oplus$ for direct sum (of matrices [15, p. 5] and of subspaces), and $\otimes$ for tensor product ($=$Kronecker product) of matrices [15, p. 8].

(c) We denote by $\mathcal{V}$ a vector space over F, usually n-dimensional, and in fact usually the space of $n\times 1$ matrices (column vectors) over F. We denote by $\mathcal{V}^*$ the $*$-dual of $\mathcal{V}$, i.e., the space of semilinear mappings from $\mathcal{V}$ into F. (In the simplic case $\mathcal{V}^*$ is just the usual dual of $\mathcal{V}$.) If $\mathcal{U}$ is a subspace of $\mathcal{V}$, we denote by $\mathcal{U}^0$ its annihilator in $\mathcal{V}^*$. If $S:\mathcal{V}\to\mathcal{V}^*$ is a linear map here, we denote by $(S\mathcal{U})^0$ the annihilator (in $\mathcal{V}=\mathcal{V}^{**}$) of the subspace $S\mathcal{U}$ of $\mathcal{V}^*$. When convenient, we regard an $n\times n$ matrix S as a linear mapping of $\mathcal{V}$ (regarded as the space of $n\times 1$ matrices) into $\mathcal{V}^*$, with corresponding $*$-bilinear ("sesquilinear") form, $(x,y)\mapsto x^*Sy$, and $*$-quadratic form, $x\mapsto x^*Sx$, on $\mathcal{V}$. In this case S^* also maps $\mathcal{V}\,(=\mathcal{V}^{**})$ into $\mathcal{V}^*$, so for example, if S is nonsingular, $S^{*-1}S$ maps $\mathcal{V}$ into $\mathcal{V}$.

We conclude this section by reducing the study of conjointness, injointness, and rejunctivity for arbitrary square matrices to the corresponding study for nonsingular matrices.

We begin with a definition, which can be satisfied even if the pencil involved has no elementary divisors (finite or infinite; for the meaning of "infinite elementary divisor" see [11, Vol. II, pp. 26–27]):

DEFINITION 1.15. A square matrix S is *totally singular* (relative to (F,E)) provided every finite elementary divisor of the pencil $S+tS^*$ is a power of t. (Thus a matrix is totally singular iff it is the matrix of a "totally degenerate" $*$-bilinear form [10, top of p. 70].)

FACT 1.16. *Let S be totally singular (relative to an admissible pair (F,E)). Then*

(a) S is rejunctive and injoint, and
(b) $\bar{S}$ and S^* are totally singular.

Proof. (a): For each m the matrix $J=J_m$ (Notation 1.14(a)) is trivially rejunctive and obviously injoint (since $G^*JG=J^*$ and $G=G^{-1}$). By [1, Corollary 4.8] each totally singular matrix is $*$-congruent to the direct sum of matrices J_m for suitable values of m and hence by Fact 1.7 is rejunctive and injoint. (b): This follows easily from (a), since F-equivalent pencils have the same elementary divisors [11, Vol. II, p. 27]; here S is rejunctive and injoint, so $\bar{S}+t\bar{S}^*=C^*(S+tS^*)C$ for some $C=\bar{C}^{-1}$ by Fact 1.2, and $S^*+tS=D^*(S+tS^*)D$ for some $D=D^{-1}$ by Definition 1.1. ■

FACT 1.17 (from [1, Corollary 4.8]). *Each square matrix S over F is $*$-congruent to the direct sum $S_0\oplus S_1$ of a totally singular matrix S_0 and a nonsingular matrix S_1, and the $*$-congruence types of S_0 and S_1 are determined by that of S.*

DEFINITION 1.18. In Fact 1.17 we shall call S_0 the *totally singular part* of S, and S_1 the *nonsingular part* of S, though of course they are defined only up to $*$-congruence (and only relative to the admissible pair (F, E), which will be understood from context wherever we use these terms).

THEOREM 1.19. *A matrix is conjoint, injoint, or rejunctive iff, respectively, its nonsingular part is conjoint, injoint, or rejunctive.*

Proof. The "if" parts are clear from Facts 1.17, 1.16(a), and 1.7. To prove the "only if" parts, we may assume (by Facts 1.17 and 1.7) that $S = S_0 \oplus S_1$ with S_0 totally singular and S_1 nonsingular. Put $T = S^*$ for the conjointness and injointness parts of the proof, and put $T = \bar{S}$ for the rejunctivity part. In either case $C^*SC = T$ for some nonsingular C (by the "only if" hypothesis plus Definition 1.1 and Fact 1.2), and $T = T_0 \oplus T_1$ conformably with $S_0 \oplus S_1$. Clearly T_1 ($= S_1^*$ or $\bar{S}_1$) is nonsingular, and T_0 ($= S_0^*$ or $\bar{S}_0$) is totally singular by Fact 1.16(b). By [1, Corollary 4.8] we thus have $T_1 = C_{11}^* S_1 C_{11}$, with C_{11} nonsingular, and with $C_{11} = C_{11}^{-1}$ if $C = C^{-1}$ and $C_{11} = \bar{C}_{11}^{-1}$ if $C = \bar{C}^{-1}$. The rejunctivity proof is now immediate from Fact 1.2, and the other proofs are immediate from Definition 1.1. ■

With Theorem 1.19 we have thus reduced the study of conjointness, injointness, and rejunctivity to the study of the nonsingular case. In the next two sections we deal exclusively with this case, deriving various conditions (necessary and/or sufficient). In Section 4 we show that rejunctivity always implies injointness and, except in certain complic cases where char $F = 2$, the converse also holds. In Section 5 we treat three special cases where, in principle, one can actually determine which matrices are conjoint, which injoint, and which rejunctive. The first of these special cases (Section 5.A) includes the usual complex case, the second (Section 5.B) is the (F, E)-neutral case, and the third (Section 5.C) is the 2×2 case, which provides several salient examples and counterexamples.

2. IMPLICATIONS INVOLVING CONJOINTNESS, INJOINTNESS, AND REJUNCTIVITY

Throughout this section (F, E) is an admissible pair of fields, arbitrary except as otherwise specified, and S and A are nonsingular matrices over F with $A = S^{*-1}S$.

LEMMA 2.1.

(a) *Always* $SAS^{-1}=A^{*-1}=S^*AS^{*-1}$.

(b) *Further,* $C^*SC=S^*$ *implies* $C^{-1}AC=A^{-1}$ *and hence implies* $(CS^{-1})^{-1}A(CS^{-1})=A^*$.

(c) *Finally,* S *rejunctive implies* A *is similar* (*to an* (E,E)*-cosquare* [2, Section 1], *in particular*) *to* A^* *and to* A^{-1}.

Proof. Routine, hence omitted. ■

In the following result the existence (of $S_1,\dots,S_k$) is just the existence part of [1, Theorem 2.3]. The uniqueness (up to $*$-congruence) was inadvertently stated there in a form different from what is needed here. We state the appropriate form of the uniqueness here. (This overlaps with [17, Theorem 16, p. 508], which gives a form of the uniqueness in the language of modules.) The proof is a routine application of [11, Vol. I, p. 220], and hence is omitted.

FACT 2.2. *Let* $p(t)$ *be the minimum polynomial of* A, *with factorization* $p(t)=q_1(t)^{m(1)}\cdots q_k(t)^{m(k)}$, *where the polynomials* $q_1(t),\dots,q_k(t)$ *are* (*monic*) $*$*-self-reciprocal, pairwise coprime in* $F[t]$, *and the exponents* $m(1),\dots,m(k)$ *are positive.*

(a) *Then* S *is* $*$*-congruent to* $S_1\oplus\cdots\oplus S_k$ *such that* $q_i(t)^{m(i)}$ *is the minimum polynomial of* $S_i^{*-1}S_i$ *for each* i.

(b) *Furthermore, if* $T_1,\dots,T_k$ *are nonsingular matrices such that* $q_i(t)^{m(i)}$ *is the minimum polynomial of* $T_i^{*-1}T_i$ *for each* i *and* $C^*(S_1\oplus\cdots\oplus S_k)C=T_1\oplus\cdots\oplus T_k$, *then* $C=C_1\oplus\cdots\oplus C_k$ *conformably to* $S_1\oplus\cdots\oplus S_k$ (*and hence* $T_i=C_i^*S_iC_i$ *for each* i).

THEOREM 2.3. *Let* $p(t)$ *be the minimum polynomial of* A, *with factorization* $p(t)=q_1(t)^{m(1)}\cdots q_k(t)^{m(k)}$, *where the polynomials* $q_1(t),\dots,q_k(t)$ *are distinct* $E[t]$*-irreducibly self-reciprocal and the exponents* $m(1),\dots,m(k)$ *are positive. Then*

(a) S *is* $*$*-congruent to* $S_1\oplus\cdots\oplus S_k$ *such that* $q_i(t)^{m(i)}$ *is the minimum polynomial of* $S_i^{*-1}S_i$ *for each* i; *and*

(b) S *is conjoint, injoint, or rejunctive iff, respectively, every* S_i *in* (a) *is conjoint, injoint, or rejunctive.*

Proof. Part (a) is a special case of Fact 2.2(a), and the "if" parts of (b) are immediate from Fact 1.7; we prove the "only if" parts. Thus let $S_1,\dots,S_k$ be as in (a); by Fact 1.7 we may assume S itself $=S_1\oplus\cdots\oplus S_k$. For each i, put $A_i=S_i^{*-1}S_i$; then $q_i(t)^{m(i)}$ is the minimum polynomial of A_i and, since it is self-reciprocal in $E[t]$, it thus must also be the minimum polynomial of A_i^{-1} and of $\bar{A}_i$ (Remark 1.11(b)). For the conjointness and injointness parts of the proof put $T_i = S_i^*$ for all i, and for the rejunctivity part put $T_i = \bar{S}_i$ for all i. In either case put $B_i = T_i^{*-1}T_i$ for all i and put $T = T_1 \oplus \cdots \oplus T_k$. Then $C^*SC = T$ by hypothesis in either case (with C nonsingular in all three parts, $C = C^{-1}$ in the injointness part, and $C = \bar{C}^{-1}$ in the rejunctivity part by Fact 1.2). Also A_i and B_i have the same minimum polynomial in either case (since $B_i = A_i^{-1}$ in the conjointness and injointness parts, and $B_i = \bar{A}_i$ in the rejunctivity part) for all i. Thus $C = C_1 \oplus \cdots \oplus C_k$ conformably to $S_1 \oplus \cdots \oplus S_k$ by Fact 2.2(b), and the rest of the proof is routine. ■

Theorem 2.3, along with Lemma 2.1 and Remarks 1.11(b) and 1.13(e), reduces the study of conjointness, injointness, and rejunctivity of nonsingular matrices S to the case where the minimum polynomial of $S^{*-1}S$ is a power $q(t)^m$ of a (monic) $E[t]$-irreducibly self-reciprocal polynomial $q(t)$ in $E[t]$. We prove the next theorem under somewhat less restrictive hypotheses, but apply it immediately (in Corollary 2.5) to just this case. (In both results we are primarily concerned with the complic case, but both also hold in the simplic case as well, parts (2) and (3) vacuously. (1) will also follow in the simplic case immediately from Theorem 4.1.)

For notation ($\hat{p}(t)$, etc.) see Notation 1.10.

THEOREM 2.4. *Let $q(t)$ be the minimum polynomial of A, and let $p(t)$ be monic in $F[t]$.*

(1) *If A is similar to A^{-1} and $q(t) = p(t)\hat{p}(t)$ with $p(t) = \bar{p}(t)$ being coprime to $\hat{p}(t)$ $(= \tilde{p}(t))$, then S is injoint.*

(2) *If A is similar to A^{-1} and $q(t) = p(t)\hat{p}(t)$ with $p(t) = \tilde{p}(t)$ being coprime to $\hat{p}(t)$ $(= \bar{p}(t))$, then S is injoint.*

(3) *If S is conjoint and $q(t) = p(t)\bar{p}(t)$ with $p(t) = \hat{p}(t)$ being coprime to $\bar{p}(t)$ $(= \tilde{p}(t))$, then S is injoint.*

Proof. (1): By [1, Corollary 2.13] and Fact 1.7 we may assume

$$S = \begin{bmatrix} 0 & I \\ M & 0 \end{bmatrix}$$

with $p(t)$ the minimum polynomial of M (and hence also of M^*). Then $A = M \oplus M^{*-1}$ and $A^{-1} = M^{-1} \oplus M^*$, so M and M^* are similar (in the complic case because A and A^{-1} are similar and M and M^{-1} have no common roots). Thus there is a nonsingular $H = H^*$ (over F) such that $M^* = H^{-1}MH$ (in the complic case by [5, Corollary], in the simplic case by

[18, Theorem 1]), so

$$\begin{bmatrix} 0 & H^{-1} \\ H & 0 \end{bmatrix}\begin{bmatrix} 0 & I \\ M & 0 \end{bmatrix}\begin{bmatrix} 0 & H \\ H^{-1} & 0 \end{bmatrix}=\begin{bmatrix} 0 & M^* \\ I & 0 \end{bmatrix}=S^*$$

is involutorily $*$-congruent to S. Therefore S is injoint.

(2): Again we may assume

$$S=\begin{bmatrix} 0 & I \\ M & 0 \end{bmatrix}$$

with $p(t)$ the minimum polynomial of M, but here M and M^{-1} are similar (because A and A^{-1} are similar and here M and M^* have no common roots). Thus $M=QP$ with Q and P involutory over F (by [8], [4, Theorem 2 plus Fact 1]) and hence S is involutorily $*$-congruent over F to

$$\begin{bmatrix} P^* & 0 \\ 0 & Q \end{bmatrix}\begin{bmatrix} 0 & I \\ M & 0 \end{bmatrix}\begin{bmatrix} P & 0 \\ 0 & Q^* \end{bmatrix}=\begin{bmatrix} 0 & M^* \\ I & 0 \end{bmatrix}=S^*,$$

so S is injoint.

(3): Here A is similar to $A_1 \oplus A_2$ with $p(t)$ and $\bar{p}(t)$ the respective minimum polynomials of A_1 and A_2. By Fact 1.7 we may assume A itself $=A_1 \oplus A_2$. By Remark 1.11(b) the minimum polynomial of A_1^* is $\bar{p}(t)$, that of A_2^{-1} is $\hat{p}(t)=p(t)$, that of A_2^* is $p(t)$, and that of A_1^{-1} is $\tilde{p}(t)=\bar{p}(t)$. Thus, since $A^*S=S^*=SA^{-1}$ and $p(t)$ and $\bar{p}(t)$ are coprime, S must $=S_1 \oplus S_2$ conformably with $A_1 \oplus A_2$ (by [11, Vol. I, p. 220]). Since S is conjoint, $C^*SC=S^*$ for some C over F, and hence $AC=CA^{-1}$ by Lemma 2.1; in partitioned form this yields

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}=AC=CA^{-1}=\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}\begin{bmatrix} A_1^{-1} & 0 \\ 0 & A_2^{-1} \end{bmatrix},$$

so $A_1C_{11}=C_{11}A_1^{-1}$ and $A_2C_{22}=C_{22}A_2^{-1}$. Thus $C_{11}=0$ because A_1 and A_1^{-1} have no common roots (again by [11, Vol. I, p. 220]), and similarly $C_{22}=0$, so $C_{12}^*S_1C_{12}=S_2^*$ because $C^*SC=S^*$. Therefore $S=S_1 \oplus S_2$ is $*$-congruent to $(C_{12}^* \oplus I)S(C_{12} \oplus I)=S_2^* \oplus S_2$, which is clearly injoint (it is $*$-congruent to $(S_2^* \oplus S_2)^*=S_2 \oplus S_2^*$ via the involutory matrix $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$). Thus S itself must be injoint (by Fact 1.7). ■

COROLLARY 2.5. *Let $q(t)$ be $E[t]$-irreducibly self-reciprocal in $E[t]$, $p(t)$ be monic in $F[t]$, and $q(t)^m$ be the minimum polynomial of A.*

(1) *If A is similar to A^{-1} and $q(t)=p(t)\hat{p}(t)$ with $p(t)=\bar{p}(t)$, then S is injoint.*

(2) *If A is similar to A^{-1} and $q(t)=p(t)\hat{p}(t)$ with $p(t)=\tilde{p}(t)$, then S is injoint.*

(3) *If S is conjoint and $q(t)=p(t)\bar{p}(t)$ with $p(t)=\hat{p}(t)$, then S is injoint.*

Proof. In each part it suffices to show that the corresponding part of Theorem 2.4 applies. The details are routine and hence omitted. ■

Concerning the stronger hypothesis in (3) [as compared with (1) and (2)] in Theorem 2.4 and Corollary 2.5, see Remark 5.15 below. In both results rejunctivity will follow from injointness by Theorem 4.7.

The following result applies the foregoing to the usual complex case and, more generally, to the complic cases where F is algebraically closed. (Again, the rejunctivity will follow here from the injointness by Theorem 4.7, and the simplic case of this result will be trivial from Theorem 4.1. In [20] the rejunctivity for the case where char $F\neq 2$ is shown directly, by a method that makes no use of Theorem 4.7.)

COROLLARY 2.6. *Suppose A^2-I is nonsingular and all roots of A are in F. Then S conjoint implies S injoint.*

Proof. Since S is conjoint, A is similar to A^{-1} and to A^* by Lemma 2.1, so Theorem 2.3 applies and tells us that it suffices to carry out the proof for the case where the minimum polynomial of A is $q(t)^m$ with $q(t)$ being $E[t]$-irreducibly self-reciprocal. Since A^2-I is nonsingular, $q(1)q(-1)\neq 0$, so we have by Remark 1.13(d) only the following three cases to consider: (1) $q(t)=(t-\alpha)(t-\alpha^{-1})$ with $\bar{\alpha}=\alpha\notin\{1,-1\}$ [where Corollary 2.5(1) applies], (2) $q(t)=(t-\beta)(t-\beta^{-1})(t-\bar{\beta})(t-\bar{\beta}^{-1})$ with $\beta, \beta^{-1}, \bar{\beta}, \bar{\beta}^{-1}$ distinct [where Corollary 2.5(1) and (2) both apply], and (3) $q(t)=(t-\gamma)(t-\bar{\gamma})$ with $\bar{\gamma}^{-1}=\gamma\notin E$ [where Corollary 2.5(3) applies]. ■

3. FURTHER IMPLICATIONS

In this section we continue our discussion of last section, but concentrate here on the cases where A^2-I is nilpotent. Throughout this section (F, E) is a complic pair of fields, S and A are $n\times n$ nonsingular over F with $A=S^{*-1}S$, and $\theta\in F$ with $\theta\neq\bar{\theta}$. [When char $F\neq 2$ we shall usually assume $\bar{\theta}=-\theta$ to save writing $\theta-\bar{\theta}$ for θ, and when char $F=2$ we shall usually assume $\theta+\bar{\theta}=1$ to save writing $(\theta+\bar{\theta})^{-1}\theta$ for θ.]

The cases where A^2-I is nilpotent are the most interesting, but in some ways (especially when char $F=2$) are also the most troublesome. We shall (when char $F\neq 2$) usually separate our considerations (as we may by Theorem 2.3) into the case where $A-I$ is nilpotent and the case where $A+I$ is nilpotent. We begin with two results which follow easily from [1, Theorem 2.4 and Lemma 2.8] plus the fact that $T\otimes D$ is $*$-congruent (by a permutation) to $D\otimes T$ for all square matrices T and D. [In each of the two results (b) is a special case of (a) but is stated separately for convenient reference.]

See Notation 1.14 for the meanings of $\otimes, G_i, J_i, G, J$.

Fact 3.1.

(a) *If $A-I$ is nilpotent, then there exist diagonal matrices $D_1, D_2, \dots, D_n$ over E (some of which may be 0×0) such that S is $*$-congruent to $S_1\oplus S_2\oplus\cdots\oplus S_n$ with*

$$S_i=[G_i(I_i+\theta J_i)]\otimes D_i$$

for each i.

(b) *When $A-I$ is similar to $J\otimes I$ here, then S is $*$-congruent to $[G(I+\theta J)]\otimes D$ for some diagonal matrix D over E.*

Fact 3.2.

(a) *If $A+I$ is nilpotent,* char $F\neq 2$, *and $\bar{\theta}=-\theta$, then there exist diagonal matrices $D_1,\dots,D_n$ over E such that S is $*$-congruent to $S_1\oplus\cdots\oplus S_n$ with*

$$S_i=[G_i(J_i+\theta I_i)]\otimes D_i$$

for each i.

(b) *When $A+I$ is similar to $J\otimes I$ here, then S is $*$-congruent to $[G(J+\theta I)]\otimes D$ for some diagonal matrix D over E.*

We shall derive conditions on the matrices S_i of Facts 3.1 and 3.2 that are necessary and/or sufficient for conjointness (or for injointness or for rejunctivity) of S. We treat first the cases where char $F\neq 2$; here the results are reasonably complete and satisfying (from an esthetic, if not a utilitarian, viewpoint).

Theorem 3.3. *Let* char $F\neq 2$ *with $\bar{\theta}=-\theta$, and let $A-\alpha I$ be similar to $J_m\otimes I$ with $\alpha\in\{1,-1\}$.*

(a) *When $(-1)^m=-\alpha$, then S is injoint (hence also conjoint) and rejunctive.*

(b) *Now suppose* $(-1)^m=\alpha$, *and let* D *be as in Fact* 3.1(b) (*if* $\alpha=1$) *or Fact* 3.2(b) (*if* $\alpha=-1$). *Then* S *is conjoint iff* θD *is conjoint*, S *is injoint iff* θD *is injoint, and* S *is rejunctive iff* θD *is rejunctive.*

Proof. Let Z be the $m\times m$ diagonal matrix

$$Z=\operatorname{diag}\left(1,-1,1,-1,\dots,(-1)^{m-1}\right).$$

Then $Z=Z^{-1}=\bar{Z}^{-1}=Z^*$, and one thus easily verifies that

$$Z^*GZ=(-1)^{m-1}G,\qquad ZJ=-JZ$$

(where $G=G_m$ and $J=J_m$), from which follow

$$\begin{aligned} Z^*\left[G(I+\theta J)\right]Z&=(-1)^{m-1}\left[G(I-\theta J)\right],\\ Z^*\left[G(J+\theta I)\right]Z&=(-1)^{m}\left[G(J-\theta I)\right]. \end{aligned}\tag{$*$}$$

These last two equations prove (a) in view of Facts 3.1(b), 3.2(b), 1.7, and 1.2 (since D is injoint and rejunctive). They are also highly useful in proving the "if" parts of (b). For, suppose $C^*(\theta D)C=(\theta D)^*$. Then $C^*DC=-D$ and so

$$\begin{aligned}(Z\otimes C)^*\left\{\left[G(I+\theta J)\right]\otimes D\right\}(Z\otimes C)&=\left\{Z^*\left[G(I+\theta J)\right]Z\right\}\otimes(C^*DC)\\ &=(-1)^m\left[G(I-\theta J)\right]\otimes D\end{aligned}$$

and likewise

$$(Z\otimes C)^*\left\{\left[G(J+\theta I)\right]\otimes D\right\}(Z\otimes C)=(-1)^{m+1}\left[G(J-\theta I)\right]\otimes D,$$

and of course $Z\otimes C=(Z\otimes C)^{-1}$ if $C=C^{-1}$, and $Z\otimes C=(\bar{Z}\otimes\bar{C})^{-1}$ if $C=\bar{C}^{-1}$.

Finally, we prove the "only if" parts of (b) for $\alpha=1$ (and m even). (The other case, where $\alpha=-1$ and m is odd, is proved similarly.) Let $S=[G(I+\theta J)]\otimes D$ and $P^*SP=S^*$ (hence $=\bar{S}$). Then we get successively

$$P^*(G\otimes D)P=G\otimes D,\qquad P^*\left[(GJ)\otimes D\right]P=-(GJ)\otimes D,$$

$$P^{-1}(J\otimes I)P=P^{-1}\left[(G\otimes D)^{-1}(GJ\otimes D)\right]P$$

$$=\left[P^*(G\otimes D)P\right]^{-1}\left[P^*(GJ\otimes D)P\right]$$

$$=(G\otimes D)^{-1}(-GJ\otimes D)=-J\otimes I,$$

$$P^*\left[(GJ^{m-1})\otimes D\right]P=P^*(G\otimes D)PP^{-1}(J\otimes I)^{m-1}P$$

$$=(G\otimes D)\left[(-1)^{m-1}(J\otimes I)^{m-1}\right]=-(GJ^{m-1})\otimes D.$$

The last equation tells us $P^*(D\oplus 0)P=-(D\oplus 0)$ [because $GJ^{m-1}=\operatorname{diag}(1,0,0,\cdots,0)$ and so $GJ^{m-1}\otimes D=D\oplus 0$]. Since D is nonsingular, P must thus be lower block-triangular conformably with $D\oplus 0$; say

$$P=\begin{bmatrix} C & 0 \\ X & Y \end{bmatrix}.$$

Then $C^*DC=-D$, and of course $C=C^{-1}$ if $P=P^{-1}$, and $C=\bar{C}^{-1}$ if $P=\bar{P}^{-1}$. Since $(\theta D)^*=-\theta D=(\overline{\theta D})$, this completes the proof. ■

THEOREM 3.4. *Let* $\operatorname{char} F\neq 2$, $\bar{\theta}=-\theta$, *and* $A-\alpha I$ *be nilpotent with* $\alpha\in\{1,-1\}$. *Let the matrices* $S_1,\cdots,S_n$ *be as in Fact* 3.1(a) (*if* $\alpha=1$) *or Fact* 3.2(a) (*if* $\alpha=-1$). *Then* S *is conjoint* [*injoint*] *iff every* S_i *is conjoint* [*injoint*].

Proof. The "if" part is immediate from Fact 1.7. We prove the "only if" part for the case $\alpha=1$. (The case $\alpha=-1$ can be treated similarly.) We may (by Fact 1.7) assume S itself $=S_1\oplus\cdots\oplus S_n$ in Fact 3.1(a) with $\bar{\theta}=-\theta$. Write $S=H+\theta K$ with $H=H^*$ and $K=K^*$. Then $H=\frac{1}{2}(S+S^*)$ is nonsingular; let $N=H^{-1}K$, so $N=\bigoplus_{i=1}^{m}J_i\otimes I$ is nilpotent (here I depends on i; it is the same size as D_i), and the coordinate vectors form a block Jordan basis for N in the space $\mathcal{V}$ of column vectors. (See Notation 1.14(c).) Suppose now that $C^*SC=S^*$. Then C and N anticommute: $C^{-1}NC=(C^*HC)^{-1}(C^*KC)=H^{-1}(-K)=-N$, so the coordinate subspaces $N^j\mathcal{V}$ and $N^{-j}0$ (=the nullspace of N^j) are C-invariant for every integer $j\geqslant 0$. Let m be a fixed positive integer ($\leqslant n$). Denote by $\mathcal{W}$ ($=\mathcal{W}_m$) the subspace

$$\mathcal{W}=(N\mathcal{V}\cap N^{-m}0)+N^{1-m}0.$$

Then $\mathcal{W}$ is a C-invariant coordinate subspace of the C-invariant coordinate subspace $N^{-m}0$, so there is a coordinate subspace $\mathcal{U}$ ($=\mathcal{U}_m$) such that $N^{-m}0=\mathcal{U}\oplus\mathcal{W}$. Let $\mathcal{B}=\{x_1,x_2,\cdots,x_{n(m)}\}$ be a coordinate basis for $\mathcal{U}$. It is not hard to see that the principal submatrix of HN^{m-1} corresponding to the coordinate subspace $\mathfrak{U}$ is just D_m, whose (i,j) entry d_{ij} $(=d_{ij}^{(m)})$ is therefore given by $d_{ij}=x_i^*HN^{m-1}x_j$. Since C maps $\mathfrak{U}\oplus\mathfrak{W}$ into itself, there are scalars c_{ij} $(=c_{ij}^{(m)})$ such that, for each i,

$$Cx_i-\sum_{j=1}^{n(m)}c_{ji}x_j\in\mathfrak{W}.$$

Let C_m be the $n(m)\times n(m)$ matrix whose (i,j) entry is c_{ij}. Since C is nonsingular, $C(\mathfrak{U}\oplus\mathfrak{W})=\mathfrak{U}\oplus\mathfrak{W}$, and hence C_m is also nonsingular (because $C\mathfrak{W}\subseteq\mathfrak{W}$). Also $C_m=C_m^{-1}$ if $C=C^{-1}$. Note that, if $y\in N\mathfrak{V}$, then $y=Nv$ for some $v\in\mathfrak{V}$, so $x^*HN^{m-1}y=x^*HN^mv=x^*N^{*m}Hv=0$ for all $x\in N^{-m}0$, and hence $x^*HN^{m-1}w=0$ for all $x\in\mathfrak{U}$ and all $w\in\mathfrak{W}$. Thus

$$\begin{aligned}(-1)^{m-1}d_{ij}&=(-1)^{m-1}x_i^*HN^{m-1}x_j=x_i^*HC^{-1}N^{m-1}Cx_j\\&=x_i^*C^*HN^{m-1}Cx_j=\Big(\sum_r c_{ri}x_r\Big)^*HN^{m-1}\Big(\sum_s c_{sj}x_s\Big)\\&=\sum_{r,s}\bar c_{ri}d_{rs}c_{sj},\end{aligned}$$

since the terms involving vectors in $\mathfrak{W}$ drop out. Therefore $C_m^*D_mC_m=(-1)^{m-1}D_m$, which, together with Theorem 3.3(b), shows that S_m is conjoint, and is injoint if S is, when m is even. Of course, S_m is injoint by Theorem 3.3(a) when m is odd. ■

REMARK 3.5. We shall not need the rejunctivity result analogous to Theorem 3.4, but it is easy to see that it holds by the same method of proof (since, in the proof, $\mathfrak{U}$ and $\mathfrak{W}$ are coordinate subspaces and $\mathfrak{B}$ is a coordinate basis for $\mathfrak{U}$, and hence $\bar Cx_i-\sum_j\bar c_{ji}x_j\in\mathfrak{W}$, so $C_m=\bar C_m^{-1}$ if $C=\bar C^{-1}$). A more roundabout proof will follow via Theorem 3.4 plus Theorem 3.11 and Corollary 4.4 below. However, the rejunctivity result, unlike the injointness result (see Remark 3.9), actually holds for char F arbitrary; a proof for $\alpha=(-1)^m$ (for even m when char $F=2$) can be based on the Witt cancellation theorem [13, p. 162] plus the same idea used in proving (i)$\Rightarrow$(ii) in Theorem 3.11.

Next we derive results for char $F=2$ corresponding to those in Theorem 3.3 for char $F\neq 2$. The more difficult results are treated separately (in Theorems 3.8 and 3.11).

THEOREM 3.6. *Let* char $F=2$, *and* $A-I$ *be similar to* $J_m \otimes I$. *Then*

(a) S *is conjoint*,
(b) S *is injoint if* $m=2$, *and*
(c) S *is injoint and rejunctive if* m *is odd.*

Proof. There is a $\theta \in F$ for which $\theta+\bar{\theta}=1$; with such a θ chosen, we may by Fact 3.1(b) assume S itself $=G(I+\theta J)\otimes D$ with $G=G_m$, $J=J_m$, and $D=D^*$ nonsingular. It clearly suffices to carry through the proof for the case where $D=[1]$, i.e., where $S=G(I+\theta J)$.

To do this, we first pick a polynomial $p(t)\in F[t]$ so that

$$t^m \text{ divides } \bar{p}(t)p(t)-(t+1)^{m-1}.$$

(This is always possible because every element of E is of the form $c+\bar{c}$ for some $c\in F$.) When $m=2$ or m is odd, we make the following particular choice for $p(t)$:

$$p(t)=1+\theta t \qquad \text{if} \quad m=2,$$

$$p(t)=(1+t)^{\frac{1}{2}(m-1)} \qquad \text{if} \quad m \text{ is odd}.$$

Next, we denote by Q $(=Q_m)$ the $m\times m$ matrix over F whose (i,j) entry is the binomial coefficient $\binom{i-1}{j-1}$. Then (see [2, Section 3]) $Q=Q^{-1}=\bar{Q}$, $(I+J)^{-1}Q=Q(I+J)$, and $GQ^*G=(I+J)^mQ$. Now we put $C=p(J)Q$ and get

$$\begin{aligned} C^*GC&=Q^*\bar{p}(J^*)Gp(J)Q=Q^*G\bar{p}(J)p(J)Q\\ &=Q^*G(I+J)^{m-1}Q=Q^*G(I+J)^mQ(I+J)\\ &=Q^*(Q^*G)(I+J)=G(I+J),\\ C^*GJC&=(C^*GC)C^{-1}JC=G(I+J)Q\left[p(J)\right]^{-1}Jp(J)Q\\ &=G(I+J)QJQ=G(I+J)\left[\left(-I+(I+J)^{-1}\right)Q\right]Q\\ &=G(-I-J+I)=-GJ=GJ \end{aligned}$$

(since $J^m=0$, $-1=1$ in F, etc.). Thus (since $\bar{\theta}=1+\theta$)

$$C^*SC=C^*G(I+\theta J)C=G(I+J+\theta J)=S^*=\bar{S},$$

so S is conjoint. To see that S is injoint and rejunctive if m is odd, note that $(I+J)^k Q$ is an involutory matrix over E for every integer k, and in particular for $k=\frac{1}{2}(m-1)$ (and hence $C=C^{-1}=\bar{C}$) when m is odd. To see that S is injoint if $m=2$, note that in the 2×2 case $Q=I+J$ and $J^2=0$, so $C^2=[(I+\theta J)Q]^2=[I+(1+\theta)J]^2=I$ here. ■

The following is immediate from Theorem 3.6(a) and Facts 3.1 and 1.7:

COROLLARY 3.7. *If* char $F=2$ *and* $A-I$ *is nilpotent, then* S *is conjoint.*

The next lemma fills part of the gap in the injointness part of Theorem 3.6, which latter makes clear the need for the complication in the hypothesis (in Lemma 3.8) about m. This hypothesis in turn corresponds to difficulties in the proof, in particular the need to consider "second-order terms" (which do not occur at all when $m=2$) even though they later cancel out (since the relevant submatrix of $X^{-1}(L_2L_0+L_0L_2)X$ is 0 whatever L_2 is).

LEMMA 3.8. *Let* char $F=2$, m *be an even integer* >2, $A-I$ *be similar to* $J_m\otimes I$, *and* D *be as in Fact* 3.1(b). *If* S *is injoint, then* D *is* $*$-*congruent to* $\begin{bmatrix}0&1\\1&0\end{bmatrix}\otimes I$.

Proof. By Fact 3.1(b) we may assume $S=G(I+\theta J)\otimes D$, where $G=G_m$, $J=J_m$, $\theta+\bar{\theta}=1$, and $D=D^*$ is nonsingular over F. Let D be $d\times d$. Then there is an $md\times md$ matrix $C=C^{-1}$ (over F) such that $C^*SC=S^*$. Let Q $(=Q_m)$ be the same $m\times m$ matrix as in the proof of Theorem 3.6. Put $L=C(Q\otimes I)$ $(=C(Q_m\otimes I_d))$; then $C=L(Q\otimes I)$.

First we show that L commutes with $J\otimes I$. Let $H=\bar{\theta}S+\theta S^*$ and $K=S+S^*$. Then $H=H^*=G\otimes D$, $K=K^*=GJ\otimes D$, and $S=H+\theta K$, so $C^*SC=S^*=H+\bar{\theta}K=H+K+\theta K=G(I+J+\theta J)\otimes D$. Thus

$$\begin{aligned}L^*(H+\theta K)L=L^*SL&=(Q\otimes I)^*C^*SC(Q\otimes I)\\&=\left[Q^*G(I+J+\theta J)Q\right]\otimes D\\&=\left\{Q^*GQ\left[(I+J)^{-1}+\theta(I+J)^{-1}J\right]\right\}\otimes D\\&=\left[Q^*GQ(I+J)^{-1}(I+\theta J)\right]\otimes D,\end{aligned}$$

since $(I+J)Q=Q(I+J)^{-1}$ and hence $JQ=Q(I+J)^{-1}J$. Therefore

$$\begin{aligned}L^{-1}(J\otimes I)L=L^{-1}(H^{-1}K)L&=(L^*HL)^{-1}(L^*KL)\\&=\left\{\left[Q^*GQ(I+J)^{-1}\right]\otimes D\right\}^{-1}\left\{\left[Q^*GQ(I+J)^{-1}J\right]\otimes D\right\}\\&=J\otimes I.\end{aligned}$$

Thus $L=\sum_{i=0}^{m-1}(J^i\otimes L_i)$ for suitable $d\times d$ matrices $L_0, L_1,\cdots,L_{m-1}$ over F.

Next we show that the following must hold:

$$L_0^*DL_0 = D = L_0^*DL_1 + L_1^*DL_0, \qquad (*)$$

$$0 = L_0^2 - I = L_0L_1 + L_1L_0 = L_2L_0 + L_0L_2 + L_1^2 + L_0L_1. \qquad (**)$$

Namely, $Q^*GQ = G(I+J)^m$ and $m \geqslant 2$, so

$$\begin{aligned}\left[G(I+J)^{m-1}\right] \otimes D &= \left[(Q^*GQ)(I+J)^{-1}\right] \otimes D = L^*HL \\ &= [I \otimes L_0 + J \otimes L_1 + \cdots]^*(G \otimes D)[I \otimes L_0 + J \otimes L_1 + \cdots] \\ &= G \otimes (L_0^*DL_0) + (GJ) \otimes (L_0^*DL_1) + (J^*G) \otimes (L_1^*DL_0) + \cdots\end{aligned}$$

(where the dots indicate terms whose first tensor factors involve higher powers of J). This proves $(*)$, since $m-1=1$ in F and $J^*G = GJ$. Also, since $m>2$ and $QJQ = J(I+J)^{-1}$, etc., we have

$$\begin{aligned}I \otimes I = C^2 &= L(Q \otimes I)L(Q \otimes I) = L\left[(Q \otimes I)\left(\sum_{i=0}^{m-1} J^i \otimes L_i\right)(Q \otimes I)\right] \\ &= L\left[(QIQ) \otimes L_0 + (QJQ) \otimes L_1 + (QJ^2Q) \otimes L_2 + \cdots\right] \\ &= L\left[I \otimes L_0 + J(I+J)^{-1} \otimes L_1 + J^2(I+J)^{-2} \otimes L_2 + \cdots\right] \\ &= \left[I \otimes L_0 + J \otimes L_1 + J^2 \otimes L_2 + \cdots\right] \\ &\quad \times \left[I \otimes L_0 + J \otimes L_1 + J^2 \otimes (L_1 + L_2) + \cdots\right] \\ &= I \otimes L_0^2 + J \otimes (L_0L_1 + L_1L_0) + J^2 \otimes \left(L_0L_1 + L_0L_2 + L_1^2 + L_2L_0\right) + \cdots,\end{aligned}$$

from which $(**)$ follows.

Next, we apply Fact 3.1(a) to $T = D[I + \theta(L_0 - I)]$. Namely, note that $D = D^*$ and by $(**)$ $L_0 = L_0^{-1}$, so by $(*)$ $DL_0 = L_0^*D = (DL_0)^*$. Also, $(L_0 - I)^2 = 0$ by $(**)$, so $(T^{*-1}T - I)^2 = 0$. Thus by Fact 3.1(a) there are nonsingular matrices X, $D_1 = D_1^*$, $D_2 = D_2^*$ over F such that

$$X^*TX = \begin{bmatrix} D_1 & 0 & 0 \\ 0 & \theta D_2 & D_2 \\ 0 & D_2 & 0 \end{bmatrix}.$$

Since $D=\bar{\theta}T+\theta T^*$ and $D(L_0-I)=T+T^*$, we thus have

$$X^*DX=\begin{bmatrix} D_1 & 0 & 0 \\ 0 & 0 & D_2 \\ 0 & D_2 & 0 \end{bmatrix}, \qquad X^{-1}(L_0-I)X=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \end{bmatrix}$$

conformably. Thus it suffices to show that D_1 is $*$-congruent to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\otimes I$. To do this, we partition $X^{-1}L_1X$ conformably to $X^{-1}L_0X$ and use the fact from $(**)$ that $L_0L_1=L_1L_0$:

$$X^{-1}L_1X=\begin{bmatrix} L_{11} & L_{12} & 0 \\ 0 & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix}.$$

Next we compute the upper left blocks of

$$X^{-1}(L_2L_0+L_0L_2+L_1^2+L_1L_0)X, \qquad X^{-1}(D^{-1}L_1^*D+L_1+L_0)X$$

and get, respectively, $L_{11}^2+L_{11}$ (whatever L_2 itself is), $D_1^{-1}L_{11}^*D_1+L_{11}+I$, both of which are consequently 0. (It is routine to infer from $(*)$ and $(**)$ that $D^{-1}L_1^*D=L_1+L_0$.) Thus $L_{11}=L_{11}^2$ and $L_{11}^*D_1=D_1(L_{11}+I)$, so there is a nonsingular matrix Y over F such that

$$Y^{-1}L_{11}Y=\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}=Y^*L_{11}^*Y^{*-1}$$

for an identity matrix I of size equal to the rank of L_{11}. Hence

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}(Y^*D_1Y)=Y^*(L_{11}^*D_1)Y=Y^*D_1(I-L_{11})Y$$

$$=(Y^*D_1Y)\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$$

(with the outside matrices partitioned conformably). Thus

$$Y^*D_1Y=\begin{bmatrix} 0 & W^* \\ W & 0 \end{bmatrix}$$

with W nonsingular, and hence Y^*D_1Y is indeed $*$-congruent to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\otimes I$. ■

REMARK 3.9. We shall see in Theorem 3.11 below that the converse of Lemma 3.8 also holds. However, for char $F=2$, the injointness part of Theorem 3.4 would fail: for any complic pair (F, E) with char $F=2$ it is easy to construct a 6×6 injoint matrix $S=S_2\oplus S_4$ in the notation of Fact 3.1(a) (with D_2 and D_4 both 1×1), and by Lemma 3.8 S_4 cannot be injoint.

The next lemma parallels, for char $F\neq 2$, the relevant part of Lemma 3.8, but its proof is much easier.

LEMMA 3.10. *Let* char $F\neq 2$, $\bar{\theta}=-\theta$, *and* $D^*=D$ *be nonsingular over* F. *If* θD *is injoint, then* D *is* $*$*-congruent to* $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\otimes I$.

Proof. Since θD is injoint and $\bar{\theta}=-\theta$, there is a matrix $C=C^{-1}$ over F such that $C^*DC=-D$. Since char $F\neq 2$, there is a nonsingular matrix P over F such that $P^{-1}CP=I_k\oplus -I_l$ for suitable nonnegative integers k and l. Partition

$$P^*DP=\begin{bmatrix} X & Y^* \\ Y & Z \end{bmatrix}$$

conformably to $I_k\oplus -I_l$; then

$$\begin{aligned}\begin{bmatrix} X & -Y^* \\ -Y & Z \end{bmatrix}&=\begin{bmatrix} I_k & 0 \\ 0 & -I_l \end{bmatrix}\begin{bmatrix} X & Y^* \\ Y & Z \end{bmatrix}\begin{bmatrix} I_k & 0 \\ 0 & -I_l \end{bmatrix}\\ &=(P^{-1}CP)^*(P^*DP)(P^{-1}CP)\\ &=P^*(C^*DC)P=-P^*DP\\ &=\begin{bmatrix} -X & -Y^* \\ -Y & -Z \end{bmatrix},\end{aligned}$$

so $X=0$, $Z=0$, and hence Y is nonsingular. Thus

$$P^*DP=\begin{bmatrix} 0 & Y^* \\ Y & 0 \end{bmatrix},$$

which is indeed $*$-congruent to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\otimes I$. ∎

Our next result fills in the gaps left in Theorem 3.6 and ties together all the injointness and rejunctivity results in Theorems 3.3 and 3.6. (For the meaning of *alternating* (or *alternate*) matrix, see [13, p. 161].)

THEOREM 3.11. *Let $A-\alpha I$ be similar to $J_m \otimes I$, $\alpha \in \{1,-1\}$, and D and θ be as in Facts* 3.1(b), 3.2(b).

(a) *If either* $\operatorname{char} F \neq 2$ *with* $\alpha=(-1)^{m-1}$ *or* $\operatorname{char} F=2$ *with m odd, then S is rejunctive and injoint.*

Now [*in* (b), (c), (d)] *suppose that* $\alpha=(-1)^m$, *or that* $\operatorname{char} F=2$ *with m even.*

(b) *The following six conditions are equivalent:*

(i) *S is rejunctive;*

(ii) *$(\theta-\bar{\theta})D$ is $*$-congruent to an alternating matrix;*

(iii) *$(\theta-\bar{\theta})D$ is $*$-congruent to* $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \otimes I$;

(iv) *D is $*$-congruent to* $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes I$;

(v) *D is $*$-congruent to* $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes I$; *and*

(vi) *S is $*$-congruent to $X \oplus X^*$ for some matrix $X=X'$ over F.*

(c) *Each condition in* (b) *implies the following condition:*

(vii) *S is injoint;*

and, unless $\operatorname{char} F=2$ *with* $m=2$, *conversely.*

(d) *Finally, if* $\operatorname{char} F \neq 2$, *then each of the seven conditions* (i)–(vii) *in* (b) *and* (c) *is equivalent to each of the following two conditions:*

(viii) *$(\theta-\bar{\theta})D$ is rejunctive; and*

(ix) *$(\theta-\bar{\theta})D$ is injoint.*

Proof. (a) is immediate from Theorems 3.3(a) and 3.6(c). In (b) the equivalence of (ii), (iii), (iv), and (v) is routine (depending only on the assumption that $\theta \neq \bar{\theta}$ and $D=D^*$ is nonsingular). Clearly (vi) implies both (i) and (vii), so (c) will follow from (b) by Lemma 3.8 (if $\operatorname{char} F=2$) and by Lemma 3.10 plus Theorem 3.3(b) (if $\operatorname{char} F \neq 2$). Of course (d) will follow from (b) and (c) by Theorem 3.3(b). Thus there remains only to show (v)$\Rightarrow$(vi) and (i)$\Rightarrow$(ii).

(v)$\Rightarrow$(vi): Let W be the $m \times m$ matrix $G(I+\theta J)$ if m is even (and hence $\alpha=1$) or the $m \times m$ matrix $G(J+\theta I)$ if m is odd (and hence $\alpha=-1 \neq 1$). Then $W=W'$, and $-W$ is $*$-congruent to W^* [by Theorem 3.6(a) if $\operatorname{char} F=2$, by the equations ($*$) in the proof of Theorem 3.3 if $\operatorname{char} F \neq 2$]. By (v) [plus Facts 3.1(b), 3.2(b)] S is $*$-congruent to

$$W \otimes \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes I \right),$$

and hence to

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes I\right) \otimes W = X \oplus -X \qquad \text{with} \quad X = I \otimes W.$$

Clearly $-X$ is $*$-congruent to X^*, so S is indeed $*$-congruent to $X \oplus X^*$ with $X = X'$.

(i)$\Rightarrow$(ii): Let $T = C^*SC$ be over E with C nonsingular.

Case 1: m is even, say $m = 2k$. Here $\alpha = 1$, and we may by Fact 3.1(b) assume $S = G(I + \theta J) \otimes D$ with $D = D^*$. In the following calculations, several simplifications arise from the fact that $(A - I)^{2k}$ and J^{2k} are zero matrices:

$$\begin{aligned}(A - I)^{2k-1} &= \left[S^{*-1}(S - S^*)\right]^{2k-1} \\ &= \left[(\theta - \bar{\theta})(I + \bar{\theta}J)^{-1}J\right]^{2k-1} \otimes I \\ &= (\theta - \bar{\theta})^{2k-1} J^{2k-1} \otimes I,\end{aligned}$$

$$\begin{aligned}(I - A^*)^{k-1}(S - S^*)(A - I)^{k-1} &= (I - A^*)^{k-1} S^*(A - I)^k \\ &= S^*(I - A^{-1})^{k-1}(A - I)^k \\ &= SA^{-k}(A - I)^{2k-1} = S(A - I)^{2k-1} \\ &= (\theta - \bar{\theta})^{2k-1}(GJ^{2k-1}) \otimes D \\ &= (\theta - \bar{\theta})^{2k-1} D \oplus 0.\end{aligned}$$

Let $\beta = (\theta - \bar{\theta})^{2k-2}$. Then $\beta \neq 0$, and $\beta(\theta - \bar{\theta})D \oplus 0$ is $*$-congruent to

$$C^*(I - A^*)^{k-1}(S - S^*)(A - I)^{k-1}C$$
$$= \left[(T - T')T^{-1}\right]^{k-1}(T - T')\left[T'^{-1}(T - T')\right]^{k-1},$$

which is an alternating matrix, so $(\theta - \bar{\theta})D$ must also be $*$-congruent to an alternating matrix (by applying the Witt cancellation theorem to the case where one direct summand is a zero matrix).

Case 2: m is odd, say $m = 2k + 1$. Here $\alpha = -1 \neq 1$, and we may assume $S = G(J + \theta I) \otimes D$ (with $D = D^*$ and $\bar{\theta} = -\theta$) by Fact 3.2(b). It will be convenient here to let $H=GJ\otimes D$ and $K=G\otimes D$, so that $S=H+\theta K$ with $H=H^*$ and $K=K^*$; also $K^{-1}H=J\otimes I$. Then $T+T'=C^*(S+S^*)C=2C^*HC$ and $T-T'=C^*(S-S^*)C=2\theta C^*KC$, and the following matrix is alternating:

$$\left[(T+T')(T-T')^{-1}\right]^k(T-T')\left[(T-T')^{-1}(T+T')\right]^k$$

$$=(T-T')\left[(T-T')^{-1}(T+T')\right]^{2k}$$

$$=2\theta^{1-2k}C^*K(K^{-1}H)^{2k}C=2\theta^{1-2k}C^*\left[GJ^{2k}\otimes D\right]C$$

$$=2\theta^{1-2k}C^*\left[D\oplus 0\right]C.$$

Thus $2\theta D=(\theta-\bar{\theta})D$ is $*$-congruent to an alternating matrix, as in case 1. ■

REMARK. Condition (iv) in Theorem 3.1(b) says just that the *Witt signature* [13, p. 170] of D, relative to (F,E), is zero.

In the next section we shall show that rejunctivity always implies injointness. The following is as much of the converse as can be easily deduced from the results of this section:

THEOREM 3.12. *Let A^2-I be nilpotent, and further, when* char $F=2$, *let A have no even-degree elementary divisors. Then S injoint implies S rejunctive.*

Proof. This is routine from Theorems 3.11(c), 3.6(c), 3.4, 2.3 and Facts 3.1(a), 3.2(a), and 1.7. ■

Before stating our next result we need a definition, which is merely a special case of a fairly standard definition in field theory:

DEFINITION 3.13. If (F,E) is complic, an (F,E)-norm is an element of E of the form $\alpha\bar{\alpha}$ with $\alpha\in F$. (We shall sometimes write "norm" instead of "(F,E)-norm" when (F,E) is understood.)

COROLLARY 3.14. *Let (F,E) be complic with E an ordered field every positive element of which is an (F,E)-norm, and suppose -1 is not an (F,E)-norm. If A^2-I is nilpotent and S is conjoint, then S is injoint and rejunctive.*

Proof. By Theorem 2.3 and Facts 3.1(a), 3.2(a), 1.7, it suffices to consider the case where $S=W\otimes D$ with $D=D^*$, $\bar{\theta}=-\theta$, and W an $m\times m$ matrix as follows: $W=G(I+\theta J)$ or $W=G(J+\theta I)$. By Theorem 3.3(a) it suffices to consider m even in the first case and m odd in the second. Then Theorem 3.3(b) applies in both cases and tells us that θD is conjoint, i.e., that D is $*$-congruent to $-D$. However, our hypotheses on (F,E) are strong enough to insure that D is $*$-congruent to $I_k\oplus -I_l$ for suitable nonnegative integers k and l, and to insure that the Sylvester inertia theorem holds. Thus here $I_k\oplus -I_l$ is $*$-congruent to $-(I_k\oplus -I_l)=-I_k\oplus I_l$, so (by the Sylvester theorem) k must equal l. Thus S is rejunctive and injoint by Theorem 3.11(b), (c). ∎

4. REJUNCTIVITY AND INJOINTNESS ARE (USUALLY) EQUIVALENT

In this section we show that rejunctivity always implies injointness and that the converse implication "usually" holds. We begin with the simplic case, where rejunctivity holds trivially. (This result was essentially proved, though not stated as such, in [20] for the case char $F\neq 2$. While this paper was in the refereeing process, the nonsingular case of Theorem 4.1 appeared in [12], where the proof is largely module-theoretic.)

THEOREM 4.1. *Let F be an arbitrary field, and let S be a square matrix over F. Then S is (F,F)-injoint (etc.; see Fact 1.4).*

Proof. By Fact 1.4 it suffices to prove that SV is symmetric for some $V=V^{-1}$ over F. By Theorem 1.19 it suffices to do this for the case where S is nonsingular, and by Theorem 2.3 and [1, Theorem 2.4] it thus suffices to do this in the two cases: (1) where $S'^{-1}S=A$ is nonderogatory, and (2) where

$$S'^{-1}S=\begin{bmatrix} X & 0\\ 0 & X\end{bmatrix} \quad\text{and}\quad S=\begin{bmatrix} P & M\\ L & Q\end{bmatrix}$$

conformably for some matrices L, M, P, Q, X with L and M nonsingular and $X=\pm I+J$. Now, A is similar to A^{-1} in case (1), so $A^{-1}=VAV$ for some matrix $V=V^{-1}$ (over F) (by [4, Theorem 2], [8]) and hence $V=AVA$. The proof is completed by proving the following two lemmas, in the second of which (Lemma 4.3) we need only the case where $X=Y=\pm I+J$. ∎

LEMMA 4.2. *Let S, A, V be square matrices over an arbitrary field F such that $S=S'A$ and $V=AVA$. Then SV is symmetric if A is nonderogatory.*

Proof. We have $SVA=S'AVA=S'V=A'SV$, so SV must be symmetric if A is nonderogatory [18, (proof of) Theorem 2]. ■

LEMMA 4.3. *Let L, M, P, Q, X, Y be square matrices over an arbitrary field F with X and Y similar nonderogatory matrices, L nonsingular, and*

$$\begin{bmatrix} P & M \\ L & Q \end{bmatrix} = \begin{bmatrix} P' & L' \\ M' & Q' \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}.$$

Then there exist involutory matrices V and W over F such that the matrix

$$\begin{bmatrix} P & M \\ L & Q \end{bmatrix} \begin{bmatrix} V & 0 \\ 0 & W \end{bmatrix}$$

is symmetric and $V=XVX$ and $W=YWY$.

Proof. Clearly M, X, Y are nonsingular (because $L=M'X$ is, etc.) and $X=M'^{-1}L$ is similar to $Y'^{-1}=LM'^{-1}$, so Y is similar to Y^{-1} (and so also X is similar to X^{-1}). Therefore $Y^{-1}=WYW$ for some $W=W^{-1}$ (over F) (by [4, Theorem 2], [8]), and so $W^{-1}=W=YWY$. Let $V=L^{-1}(YW)'L$. The rest of the proof is a routine computation using Lemma 4.2. We omit details. ■

Note that Fact 1.9 gives an obvious corollary to Theorem 4.1.

For the rest of this section we concentrate on the complic case, though some of the results hold just as well in the simplic case. We continue with an obvious corollary of Theorem 4.1.

COROLLARY 4.4. *Rejunctivity always implies injointness.*

Proof. Let (F, E) be admissible and let S be an (F, E)-rejunctive matrix. Then there is a matrix T over E which is (F, E)-congruent to S, and by Theorem 4.1 T is (E, E)-injoint, so T is also (F, E)-injoint, and by Fact 1.7 so is S. ■

Next we consider the converse of Corollary 4.4. We have seen in Theorems 3.6(b) and 3.11(b) that this converse is false in the complic case where $\operatorname{char} F=2$ and the pencil $S+tS^*$ has an elementary divisor which is a power of $(t-1)^2$. We shall see in Corollary 4.8 below that the converse is true in all other cases. Most of the proof relies on the following lemma, which we state and prove in terms of mappings (see Notation 1.14(c)).

LEMMA 4.5. *Let (F, E) be complic, let $S:\mathcal{V}\to\mathcal{V}^*$ and $C:\mathcal{V}\to\mathcal{V}$ be nonsingular linear maps with $C^*SC=S^*$ and $C=C^{-1}$, let $q(t)$ be $E[t]$-irreducibly self-reciprocal in $E[t]$ with $q(1)q(-1)\neq 0$, and let $\varepsilon\in\{1,-1\}$. Denote $S^{*-1}S$ by A, and suppose that some power of $q(A)$ is 0. Then*

(a) *there is a vector $e\in\mathcal{V}$ such that $Ce=\varepsilon e$ and such that the A-cyclic subspace $\mathcal{U}$ generated by e satisfies $(S\mathcal{U})^0\cap\mathcal{U}=0$; also*

(b) *(for any such subspace $\mathcal{U}$) we have $\mathcal{V}=\mathcal{U}\oplus(S\mathcal{U})^0$ with $A\mathcal{U}=\mathcal{U}=C\mathcal{U}$ and $A(S\mathcal{U})^0=(S\mathcal{U})^0=C(S\mathcal{U})^0=(S^*\mathcal{U})^0$.*

Proof. Part (b) follows routinely from (a) plus the hypotheses (which imply $A^{*-1}S=SA$, $CA=A^{-1}C$, etc.). In the following proof of (a) we shall write C for εC everywhere to increase readability.

First we "construct" a suitable vector e. Let m be the positive integer such that $q(A)^m=0\neq q(A)^{m-1}$. Since $q(1)q(-1)\neq 0$, the map A^2-I is thus nonsingular and, since also $q(t)$ is self-reciprocal in $E[t]$, $q(t)=t^k h(t+t^{-1})$ for some positive integer k and some $h(t)\in E[t]$. (Remark 1.13(c) is relevant here.) Let $f(t)=h(t+t^{-1})^{m-1}$. Then $f(A)$ commutes with C (because $A+A^{-1}$ does: $AC=CA^{-1}$, etc.). Also $A-CAC=A-A^{-1}=A^{-1}(A^2-I)$ is nonsingular and $f(A)=A^{-k(m-1)}q(A)^{m-1}\neq 0$, so

$$0\neq(A-CAC)f(A)=\big[(I+C)CA-CA(I+C)\big]f(A)$$
$$=\big[(I+C)f(A)\big]CA-CA\big[(I+C)f(A)\big],$$

and hence $(I+C)f(A)\neq 0$. Further, $I+A^{-1}=A^{-1}(A+I)$ is nonsingular, so

$$0\neq(I+A^{-1})(I+C)f(A)=(I+A^{-1}C)(I+C)f(A)$$
$$=(I+CA)f(A)(I+C).$$

Denote by $\mathcal{N}$ the nullspace of $I-C$. Then $(I+C)\mathcal{V}\subseteq\mathcal{N}$, so $(I+CA)f(A)\mathcal{N}\neq 0$. However, $(I+CA)(I-CA)=I-CACA=0$, so $f(A)\mathcal{N}$ is not a subset of

$$(I-CA)\mathcal{V}=(I-A^{-1}C)\mathcal{V}=S^{-1}(S-S^*C)\mathcal{V}=S^{-1}(S-C^*S)\mathcal{V}$$
$$=S^{-1}(I-C^*)S\mathcal{V}=S^{-1}(I-C)^*\mathcal{V}^*=S^{-1}\mathcal{N}^0,$$

and hence $Sf(A)\mathfrak{N} \not\subseteq \mathfrak{N}^0$. Therefore the $\mathfrak{N}$-restriction of the $*$-bilinear (or "sesquilinear") form $x^*Sf(A)y$ is nonzero, and thus (since (F,E) is complic) so is the $\mathfrak{N}$-restriction of the $*$-quadratic form $x^*Sf(A)x$. Thus there is a vector $e\in\mathfrak{N}$ (that is, $e=Ce$) such that $e^*Sf(A)e\neq 0$. Let $\mathfrak{U}$ be the A-cyclic subspace generated by e.

Next, we show that $(S\mathfrak{U})^0\cap\mathfrak{U}=0$. For this it suffices to assume $[s(A)e]^*S\mathfrak{U}=0$ with $s(t)\in F[t]$ and to show that then $q(t)^m$ divides $s(t)$. We first show $[\bar{s}(A)e]^*S\mathfrak{U}=0$. Recall that $CA=A^{-1}C$, $A^{*-1}S=SA$, etc. We have

$$\begin{aligned}
0&=\left[s(A)e\right]^*SA^ie=\left[s(A)e\right]^*C^*S^*CA^ie\\
&=\left[s(A^{-1})Ce\right]^*S^*A^{-i}Ce=\left[s(A^{-1})e\right]^*S^*A^{-i}e\\
&=\left[e^*A^{*-i}Ss(A^{-1})e\right]^*=\left[e^*s(A^*)SA^ie\right]^*\\
&=\left[e^*\bar{s}(A)^*SA^ie\right]^*=\left\{\left[\bar{s}(A)e\right]^*SA^ie\right\}^*
\end{aligned}$$

for every integer i, so $[\bar{s}(A)e]^*S\mathfrak{U}=0$. Now let $r(t)=\gcd\{s(t),\bar{s}(t)\}$. Then $r(t)\in E[t]$, and

$$r(t)=u(t)s(t)+v(t)\bar{s}(t)$$

for suitable polynomials $u(t)$, $v(t)$ in $F[t]$. Then $\bar{u}(A^{-1})\mathfrak{U}\subseteq\mathfrak{U}$ and $\bar{v}(A^{-1})\mathfrak{U}\subseteq\mathfrak{U}$, and hence

$$0\supseteq\left[s(A)e\right]^*S\bar{u}(A^{-1})\mathfrak{U}=\left[u(A)s(A)e\right]^*S\mathfrak{U},$$

$$0\supseteq\left[\bar{s}(A)e\right]^*S\bar{v}(A^{-1})\mathfrak{U}=\left[v(A)\bar{s}(A)e\right]^*S\mathfrak{U},$$

so by adding the outside inclusions we get $0=[r(A)e]^*S\mathfrak{U}$. Thus by [1, Lemma 2.6(2)] $q(t)^m$ divides $r(t)$ and hence divides $s(t)$. ∎

LEMMA 4.6. *Let (F,E) be complic, let $q(t)$ be $E[t]$-irreducibly self-reciprocal in $E[t]$ with $q(1)q(-1)\neq 0$, and let S, C, A be nonsingular matrices over F with $C^*SC=S^*$, $C=C^{-1}$, $A=S^{*-1}S$, and some power of $q(A)$ equal to 0. Then there is a nonsingular matrix D over F such that $D^*SD=S_1\oplus\cdots\oplus S_k$ and $D^{-1}CD=C_1\oplus\cdots\oplus C_k$ conformably, with $S_i^{*-1}S_i$ nonderogatory and having a cyclic (column) vector in* nullspace$(I-C_i)$ *and a cyclic vector in* nullspace$(I+C_i)$ *for each $i\in\{1,2,\cdots,k\}$.*

Proof. This is an obvious induction (on the order of S, C, A), using Lemma 4.5 (restated in terms of matrices rather than mappings). ■

THEOREM 4.7. *Let (F,E) be an arbitrary complic pair, S be (F,E)-injoint, and the pencil $S+tS^*$ have no elementary divisors divisible by $t-1$ or $t+1$. Then S is (F,E)-rejunctive.*

Proof. Let S_1 be the nonsingular part of S (Definition 1.18) defined relative to (F,E). Then [11, Vol. II, p. 39] the elementary divisors divisible by $t-1$ and $t+1$ are the same for the two pencils $S+tS^*$ and $S_1+tS_1^*$, and so by Theorem 1.19 it suffices to assume (besides the given hypotheses) that $S=S_1$, i.e., that S is itself nonsingular.

Thus let S be nonsingular, $A=S^{*-1}S$, and A^2-I be nonsingular. Since S is injoint, A is similar to A^{-1} and to A^* by Lemma 2.1. By Theorem 2.3 it thus suffices to carry out the proof for the case where the minimum polynomial of A is a power of an $E[t]$-irreducibly self-reciprocal polynomial. Finally, Lemma 4.6 reduces the proof to that for the case where A is nonderogatory with a cyclic (column) vector $e=Ce$ and $C^*SC=S^*$ with $C=C^{-1}$. (Recall that $CA=A^{-1}C$, etc.) In the A-cyclic basis generated by e, the matrix of S must be over E because

$$\begin{aligned}\left[(A^ie)^*S(A^je)\right]^* &= (A^je)^*S^*(A^ie)=(A^je)^*C^*SC(A^ie)\\ &=(CA^je)^*S(CA^ie)=(A^{-j}Ce)^*S(A^{-i}Ce)\\ &=(A^{-j}e)^*S(A^{-i}e)=(A^ie)^*S(A^je)\end{aligned}$$

for all integers i and j. Thus S is rejunctive. ■

COROLLARY 4.8. *Let (F,E) be an arbitrary complic pair and let S be a square matrix over F; in case* char $F=2$, *further let the pencil $S+tS^*$ have no even-degree elementary divisors divisible by $t-1$. Then S injoint implies S rejunctive.*

Proof. This is routine from Theorems 1.19, 2.3, 3.12, and 4.7. ■

5. THREE SPECIAL CASES

In this section we treat conjointness, etc., in three special complic cases where explicit criteria can be given.

A. The Usual Complex Case and Certain Generalizations

These are the complic pairs (F, E) where E is real-closed (and F is its algebraic closure). (According to a result [14, Theorem 56] of Artin and Schreier, these are the *only* complic pairs (F, E) for which F is algebraically closed.)

The following theorem confirms the finite-dimensional case of two conjectures of Choi [6, (iii) and (iv), p. 419]. We state it for the case that E is real-closed, but it would hold for somewhat more arbitrary complic pairs (F, E) and a particular matrix S over F if every elementary divisor of the pencil $S+tS^*$ were assumed to be a power of a linear polynomial in $F[t]$; e.g., it would hold then if E were ordered and every positive element of E were an (F, E)-norm and -1 not an (F, E)-norm.

THEOREM 5.1. *Let (F, E) be a complic pair with E real-closed (i.e., with E ordered and F algebraically closed). Then every (F, E)-conjoint matrix is (F, E)-injoint.*

Proof. By Theorem 1.19 it suffices to consider nonsingular matrices. By Theorem 2.3 it thus suffices to consider matrices S such that the minimum polynomial of $A=S^{*-1}S$ is a power of an $E[t]$-irreducibly self-reciprocal polynomial in $E[t]$. However, we have already seen (in Corollaries 2.6 and 3.14) that, in every such case where our hypothesis on (F, E) holds, S conjoint implies S injoint. ∎

REMARK 5.2. By Theorem 5.1, plus Corollaries 4.4 and 4.8, the three properties, conjointness, injointness, and rejunctivity, are equivalent for complic pairs (F, E) for which E is real-closed. An effective criterion for conjointness (and hence for injointness and rejunctivity) is known here; it can be considered a special case of one given for (F, E)-congruence of matrices over F for such pairs (F, E) in [16] (which is stated for the case where E is the real field, but holds equally well for every real-closed field E). In view of Remark 1.6, this also gives an effective criterion for involutory $*$-congruence of pairs (H, K) of $*$-symmetric matrices in the case of such complic pairs (F, E).

B. The Neutral Case (for Arbitrary Complic Pairs)

A square matrix S over F is called (F, E)-*neutral* provided it is (F, E)-congruent to a matrix in partitioned form $\begin{bmatrix} 0 & N \\ M & 0 \end{bmatrix}$ with the zero blocks on the diagonal being square [1, Definition 2.9]. There is in general no simple criterion for (F, E)-neutrality, but for matrices known to be (F, E)-neutral there is a reasonably simple effective criterion for conjointness, injointness, and rejunctivity (and again here these three properties are equivalent to each other), as given in part (c) of the following result.

THEOREM 5.3. *Let (F, E) be complic and S be a square matrix over F. Then* (a) *each of the first five of the six statements* (i)–(vi) *below implies the next;* (b) *all six statements are true if S is totally singular; and* (c) *all six are equivalent to each other if S is (F, E)-neutral:*

(i) *S is (F, E)-rejunctive;*
(ii) *S is (F, E)-injoint;*
(iii) *S is (F, E)-conjoint;*
(iv) *the pencils $S+tS^*$ and S^*+tS are (F, E)-congruent;*
(v) *the pencils $S+tS^*$ and S^*+tS are F-equivalent;*
(vi) *the pencils $S+tS^*$ and S^*+tS are $F[t]$-equivalent.*

Proof. (a): This is routine: e.g., (i)$\Rightarrow$(ii) is from Corollary 4.4, and (iii)$\Leftrightarrow$(iv) because $C^*SC=S^*$ iff $C^*(S+tS^*)C=S^*+tS$. (b) follows from (a) and Fact 1.16(a). To prove (c), it suffices (in view of (a)) to assume S is (F, E)-neutral and satisfies (vi), and then show S must be (F, E)-rejunctive. Since (vi) and (i) are invariant under (F, E)-congruence, it suffices by Theorem 1.19 and [1, Corollary 4.11] to assume S is nonsingular. Then by [1, Remark 4.10] S satisfies (v), and hence by [1, Corollary 4.12] satisfies (iv) and (iii). Thus $A=S^{*-1}S$ is similar to A^{-1} by Lemma 2.1, and by Theorem 2.3 and [1, Theorem 2.11(e)] it suffices (for the proof) to assume the minimum polynomial of A is $q(t)^m$ with $q(t)$ $E[t]$-irreducibly self-reciprocal in $E[t]$. Then by [1, Theorem 2.11(a),(b),(c)] we may assume

$$S=\begin{bmatrix} 0 & I \\ M & 0 \end{bmatrix},$$

where the minimum polynomial of M is $p(t)^m$ for some monic $p(t)$ in $F[t]$. Since $A=S^{*-1}S=M\oplus M^{*-1}$, we have $q(t)=\operatorname{lcm}\{p(t), \hat{p}(t)\}$, and also (by Remark 1.13(d))

$$q(t)=\operatorname{lcm}\{f(t), \bar{f}(t), \tilde{f}(t), \hat{f}(t)\}$$

for some monic $F[t]$-irreducible $f(t)$. We consider four cases; in the first two we show that Corollary 2.5(1) or (2) applies, i.e., that $q(t)=g(t)\hat{g}(t)$ with $g(t)\in\{\bar{g}(t), \bar{g}(t)\}$, and in the last two we show that M is similar to M^*, whence S is rejunctive as in the proof of Theorem 2.4(1).

Case 1: $q(t)=f(t)\bar{f}(t)\hat{f}(t)\tilde{f}(t)$. Here $q(t)=g(t)\hat{g}(t)$ with $g(t)=f(t)\bar{f}(t)=\bar{g}(t)$ or, equally well, with $g(t)=f(t)\tilde{f}(t)=\bar{g}(t)$.

Case 2: $q(t)=f(t)\hat{f}(t)$. Here $f(t)\in\{\bar{f}(t), \tilde{f}(t)\}$, so we can take $g(t)=f(t)$.

Case 3: $q(t)=f(t)$. Here $p(t)=q(t)=\bar{p}(t)$ is $F[t]$-irreducible, so M is similar to M^* by Remark 1.11(b).

Case 4: $q(t)=f(t)\bar{f}(t)$ and $f(t)=\hat{f}(t)$. Here $p(t)=f(t)\bar{f}(t)$, so each invariant factor $h(t)$ of M has the form $h(t)=f(t)^i\bar{f}(t)^j$ (for suitable nonnegative integers i and j) and hence satisfies $h(t)=\hat{h}(t)$ (by Remark 1.11(a)); thus M is similar to M^{*-1} (by Remark 1.11(b)), and hence M is also similar to M^* because $A=M\oplus M^{*-1}$ is similar to $A^{-1}=M^{-1}\oplus M^*$. (In case 4 we could also get S injoint by Corollary 2.5(3) and hence rejunctive by Theorem 4.7.) ■

C. *The 2×2 Case (for Arbitrary Complic Pairs)*

In this subsection we characterize conjointness, injointness, and rejunctivity of 2×2 matrices in the complic cases in terms of (complic) norms (Definition 3.13). This will thereby completely characterize these three matrix concepts in the 2×2 case for complic pairs (F, E) in (at least) the following three cases: (1) where E is real-closed (here the (F, E)-norms are just the nonnegative elements of E), (2) where E is finite (here every element of E is an (F, E)-norm), and (in principle) (3) where E is a finite extension of the rational field or of one of the p-adic fields. In any case, this will give us some insight into how these matrix concepts depend on (F, E)-arithmetic and will provide us with some salient examples.

We begin with a statement for the 1×1 case, whose proof is trivial:

FACT 5.4. *Let (F, E) be complic and $s\in F$.*

(a) *Here $[s]$ is rejunctive iff $[s]$ is injoint iff $s\in E$;*

(b) *on the other hand, $[s]$ is conjoint iff either $s\in E$ or else $-1=\bar{s}^{-1}s$ and is an (F, E)-norm.*

We next state the result for $n\times n$ matrices of rank $\leqslant 1$; we omit details of the proof, which is easily derived from the fact that every $n\times n$ matrix over F is (F, E)-congruent to a triangular matrix if (F, E) is complic [3, pp. 81–82].

FACT 5.5. *Let (F, E) be complic, and let S be an $n\times n$ matrix of rank $\leqslant 1$ over F. Then S is injoint iff S is rejunctive. Furthermore,*

(a) *if S and S^* are not proportional, then S is rejunctive (and conjoint);*

(b) *if S and S^* are proportional, then S is rejunctive [conjoint] iff every diagonal entry, considered as a 1×1 matrix over F, is rejunctive [conjoint].*

As noted earlier, every matrix $S=S^*$ is rejunctive, injoint, and conjoint. From now on, we consider nonsingular 2×2 matrices $S\neq S^*$. Here the easiest case, which must be treated by itself, is where S and S^* are proportional:

THEOREM 5.6. *Let (F,E) be complic, and let S be a nonsingular 2×2 matrix over F with S and S^* proportional but not equal.*

(i) *If S is conjoint (or rejunctive), then $S^*=-S$ (and hence* $\operatorname{char} F\neq 2$).

(ii) *Now let $S^*=-S$ $(\neq S)$.* (a) *Here S is rejunctive iff S is injoint iff* $\det S$ *is an (F,E)-norm;* (b) *on the other hand, S is conjoint iff* $\det S$ *is the sum of two (F,E)-norms.*

Proof. The proof of (i) is routine, as is the proof of (ii) through use of the fact [3, pp. 81–82] that S is (F,E)-congruent to a triangular matrix (which here must be diagonal). It also helps in proving (ii)(b) to note that $\det S$ is the difference of two norms, and hence is the sum of two norms if -1 is a norm. [Part (ii)(b) and its proof are valid even if $\operatorname{char} F=2$.] ■

COROLLARY 5.7. *In Theorem 5.6 let E be finite [E be real-closed]. Then the following four statements are equivalent:* (i) *S is rejunctive,* (ii) *S is injoint,* (iii) *S is conjoint, and* (iv) $\det S$ *is an element [a positive element] of E.*

Finally, we treat nonsingular 2×2 matrices S for which S and S^* are not proportional, i.e., for which $A=S^{*-1}S$ is nonscalar (and is thus similar to a companion matrix). By Lemma 2.1 we may as well assume A is similar to A^{-1} (and hence to A^*, and hence $\det A=\pm1$ and $\operatorname{tr} A\in E$), as we shall. The result is the following (in which the root ϕ of A need not be in F but is at worst quadratic over F):

THEOREM 5.8. *Let (F,E) be complic, S be nonsingular 2×2 over F, $A=S^{*-1}S$ be nonscalar and similar to A^{-1},* $\det(A-\phi I)=0$, *$\theta\in F$ but $\notin E$, and $\psi=\bar\theta\phi+\theta\phi^{-1}$.*

(i) *When* $\det A=-1\neq1$, *then* $\det(A-I)=0=\det(A+I)$, *and S is neither (F,E)-injoint nor (F,E)-rejunctive, and is (F,E)-conjoint iff -1 is an (F,E)-norm.*

(ii) *Now let* $\det A=1$. *Then* (a) *S is (F,E)-rejunctive iff* $\det(S-S^*)$ *is a nonzero (F,E)-norm;* (b) *S is (F,E)-injoint iff either S is (F,E)-rejunctive or $\phi=1=-1$;* (c) *when $\phi\neq1$, S is (F,E)-conjoint iff S is $(F(\psi),E(\psi))$-rejunctive;* (d) *when $\phi=1$, S is (F,E)-conjoint iff -1 is an (F,E)-norm [and is (F,E)-injoint iff $-1=1$, and is not (F,E)-rejunctive];* (e) *when $\phi\in E$ but*

$\neq 1$, *S is (F, E)-rejunctive; and* (f) *when $\phi \in F$ but $\notin E$, S is (F, E)-rejunctive iff S is (F, E)-conjoint iff* $\det S$ *is an (F, E)-norm.*

Proof. (The proof in full detail would be too long, so we omit many details.) (ii)(e) follows from Theorem 3.3(a) (when $\phi = -1 \neq 1$) and Corollary 2.5(1) (when $\phi^{-1} \neq \phi \in E$), and (ii)(f) will follow from (ii)(a) plus Corollaries 2.5(3), 4.4, and 4.8, once we note that here

$$\det(S - S^*) = (\det S^*)\det(A - I)$$

$$= (\det S)(\phi - 1)(\bar{\phi} - 1), \qquad (*)$$

since $\det A = 1$ (so $\det S = \det S^*$) and A is similar to A^* (so $\phi^{-1} = \bar{\phi}$). Thus it suffices to prove (i) and (ii)(a)–(d). First we prove three lemmas, in the first two of which we use the following notation, taken over from the usual complex case [3, p. 85]: for an $n \times n$ matrix S over F we denote by $\Gamma(S)$ the subset (depending on the pair (F, E) as well as the matrix S) of F given by

$$\Gamma(S) = \{x^*Sx : x \text{ is } n \times 1 \text{ over } F\}. \qquad \blacksquare$$

LEMMA 5.9. *Under the hypotheses of Theorem 5.8 there is always an element g in F satisfying*

$$g \in \Gamma(S) \qquad \textit{with} \qquad \det(gS^* - \bar{g}S) \neq 0. \qquad (**)$$

Furthermore, in (i) (*i.e.,* $\det A = -1 \neq 1$) *the matrix A is similar to the companion matrix* $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ *and to the diagonal matrix* $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, *and S is correspondingly $*$-congruent to the matrices*

$$\begin{bmatrix} g & \bar{g} \\ \bar{g} & g \end{bmatrix} \quad \textit{and} \quad \frac{1}{2}\begin{bmatrix} g + \bar{g} & 0 \\ 0 & g - \bar{g} \end{bmatrix}$$

for each g satisfying $(**)$ (and only for such g). Likewise, in (ii) (*i.e.,* $\det A = 1$) *the matrix A is similar to the companion matrix* $\begin{bmatrix} 0 & -1 \\ 1 & e \end{bmatrix}$ (*where* $e = \operatorname{tr} A$), *and S is correspondingly $*$-congruent to*

$$\begin{bmatrix} g & eg - \bar{g} \\ \bar{g} & g \end{bmatrix}$$

for each g satisfying $(**)$ (and only for such g).

Proof. The proof is routine, except for the existence of g satisfying $(**)$. To prove this, we may assume S is lower triangular (by [3, pp. 81–82]), say

$$S=\begin{bmatrix} a & 0 \\ c & d \end{bmatrix}.$$

Then $ad\neq 0$ and $a\in\Gamma(S)$. If $\det(aS^*-\bar{a}S)\neq 0$ here, then we can take $g=a$, so suppose $\det(aS^*-\bar{a}S)=0$. Then $c=0$, so we can take $g=a+d=[1\quad 1]S[1\quad 1]^*\in\Gamma(S)$, because $A=\operatorname{diag}(\bar{a}^{-1}a,\bar{d}^{-1}d)$ is nonscalar and hence

$$(a+d)\begin{bmatrix} \bar{a} & 0 \\ 0 & \bar{d} \end{bmatrix}-(\bar{a}+\bar{d})\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}=\begin{bmatrix} d\bar{a}-\bar{d}a & 0 \\ 0 & a\bar{d}-\bar{a}d \end{bmatrix}$$

is nonsingular. ∎

The next lemma treats rejunctivity. In it we use the following additional notation: $\Gamma_1(S)$ is the subset of $\Gamma(S)$ given by $\Gamma_1(S)=\{x^*Sx\in\Gamma(S): x\neq 0\}$.

LEMMA 5.10. *Under the hypotheses of Theorem* 5.8(ii) *the following four statements are equivalent*:

(1) S *is* (F,E)*-rejunctive*;
(2) $\det(S-S^*)\neq 0$ *and* $E\cap\Gamma(S)\neq\{0\}$;
(3) $\det(S-S^*)\neq 0$ *and* $E\cap\Gamma_1(S)$ *is nonempty*; *and*
(4) $\det(S-S^*)$ *is a nonzero* (F,E)*-norm.*

Proof. (1)⇒(4): In (1) the matrix $S-S^*$ is $*$-congruent to a nonzero alternating matrix over E, whose determinant must thus be the square of a nonzero element of E.

(4)⇒(3): Let

$$S=\begin{bmatrix} a & b \\ c & d \end{bmatrix},\qquad \alpha\in F,$$

and x^* be the 1×2 matrix $[\alpha+c-\bar{b}\ \ \bar{a}-a]$. Direct calculation then gives

$$x^*Sx-\overline{(x^*Sx)}=x^*(S-S^*)x=(a-\bar{a})\left[\alpha\bar{\alpha}-\det(S-S^*)\right].$$

Thus, if $a\neq\bar{a}$ and $\det(S-S^*)=\alpha\bar{\alpha}$, then $x^*Sx\in E\cap\Gamma_1(S)$. However, if $a=\bar{a}$, then $x^*Sx\in E\cap\Gamma_1(S)$ if $\alpha=\bar{b}-c+1$.

(3)⇒(2): Suppose $x^*Sx=0$ with $x\neq 0$. Then we may assume

$$S=\begin{bmatrix} a & 1 \\ c & 0 \end{bmatrix} \quad \text{with} \quad c\neq 0.$$

Thus $c\neq 1$ [because $\det(S-S^*)\neq 0$] and $c=-(\det S)\in E$ [because $\det A=1$], so $y^*Sy=(1-c)(a+\bar{a})$ is in $E\cap\Gamma(S)$, where $y^*=[1-c \;\; a]$. If $a+\bar{a}=0$, then $c\neq -1$ (because $S\neq -S^*$) and hence $0\neq(1-c)(1+c)=z^*Sz$ is in $E\cap\Gamma(S)$, where $z^*=[1-c \;\; 1+a]$.

(2)⇒(1): Let $0\neq g\in E\cap\Gamma(S)$. Then $g=\bar{g}$, so $\det(gS^*-\bar{g}S)=g^2\det(S^*-S)\neq 0$, and hence by Lemma 5.9 S is $*$-congruent to $\begin{bmatrix} g & cg-g \\ g & g \end{bmatrix}$, which is over E. ■

Lemma 5.11. *Under the hypotheses of Theorem* 5.8(ii), *suppose* $\phi\notin F$, *and let* $K=F(\psi)$ *and* $L=E(\psi)$. *Then* $F(\phi)=K$, *the pair* (K,L) *is complic, and the following five statements are equivalent*:

(1) S *is* (F,E)*-conjoint*;
(2) S *is* (K,L)*-conjoint*;
(3) S *is* (K,L)*-rejunctive*;
(4) $\det(S-S^*)$ *is a nonzero* (K,L)*-norm*; *and*
(5) $\det S$ *is a* (K,L)*-norm*.

Proof. It is routine to show that $F(\phi)=K$ and that (K,L) is complic, and then (1)⇒(2) and (by Lemma 5.10) (3)⇒(4) are obvious consequences. In order to show (2)⇒(3) and (4)⇒(5)⇒(1), we introduce (K,L)-conjugation, which extends (F,E)-conjugation and for which we use the same notation $(u\mapsto\bar{u})$:

$$\overline{(a+b\phi)}=\bar{a}+\bar{b}\phi^{-1} \qquad \text{for all} \quad a,b\in F.$$

We shall also find it convenient to have a notation for (K,F)-conjugation $(u\mapsto\tilde{u}=u^{\tilde{}})$ and for $(K,E(\phi))$-conjugation $(u\mapsto\hat{u}=u^{\hat{}})$:

$$(a+b\phi)^{\tilde{}}=a+b\phi^{-1}, \qquad (a+b\phi)^{\hat{}}=\bar{a}+\bar{b}\phi$$

for all $a,b\in F$. (Thus these three conjugations commute, and each is the composite of the other two.) In particular, $\bar{\phi}=\phi^{-1}=\tilde{\phi}$ $(\neq\phi)$, so (2)⇒(3) follows by applying (to (K,L)) Corollary 2.5(3) and Theorem 4.7. Also (4)⇒(5) follows by applying to (K,L) the computation $(*)$ at the beginning of the proof of Theorem 5.8.

To prove (5)$\Rightarrow$(1), let $e=\operatorname{tr}A$. Then $\phi+\phi^{-1}=e\in E$, and by Lemma 5.9 we may assume

$$S=\begin{bmatrix} g & eg-\bar{g} \\ \bar{g} & g \end{bmatrix}$$

for some nonzero $g\in F$. Let $\kappa=\bar{g}-g\phi$. Then $\det S=g^2+\bar{g}^2-e\bar{g}g=\kappa\tilde{\kappa}$, so by (5) there is an $\alpha\in K$ such that

$$\alpha\tilde{\alpha}=\kappa\tilde{\kappa}\left[(\kappa\tilde{\kappa})^{\bar{}}\right]^{-1}=\kappa\hat{\kappa}^{-1}=\bar{\kappa}\tilde{\kappa}^{-1}.$$

In order to show (1), it suffices here to show that S and $\bar{S}$ are (F,E)-congruent, since

$$S^*=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\bar{S}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Thus let

$$C=(\phi-\tilde{\phi})^{-1}\begin{bmatrix} \tilde{\phi}\tilde{\alpha}-\phi\alpha & \tilde{\alpha}-\alpha \\ \alpha-\tilde{\alpha} & \tilde{\phi}\alpha-\phi\tilde{\alpha} \end{bmatrix}.$$

Then $\tilde{C}=C$, so C is over F, and it can be verified by direct calculation that $C^*SC=\bar{S}$. However, this calculation can be simplified somewhat by first verifying that $C=BD\bar{B}^{-1}$ (or, more easily, that $C\bar{B}=BD$) with

$$B=\begin{bmatrix} 1 & -\phi \\ -\phi & 1 \end{bmatrix},\qquad D=\phi^{-1}\begin{bmatrix} 0 & \tilde{\alpha} \\ \alpha & 0 \end{bmatrix},$$

then calculating $B^*SB=(\phi-\phi^{-1})\operatorname{diag}(\kappa,-\tilde{\kappa})$, taking (K,L)-conjugates in the latter equation to get $\bar{B}^*\bar{S}\bar{B}$, and, finally, verifying that $D^*(B^*SB)D=\bar{B}^*\bar{S}\bar{B}$. ■

Proof of Theorem 5.8 (continued and completed). The parts of (i) not covered by Lemma 5.9 are routine. (ii): Here (a) is part of Lemma 5.10. The "if" part of (b) follows from Corollary 4.4 plus the fact that $\phi=1=-1$ makes Theorem 3.6(b) apply. The "only if" part of (b) follows from Corollary 4.8. Part (c) is part of Lemma 5.11 when $\phi\notin F$. When $\bar{\phi}\neq\phi\in F$, part (c) is routine from Corollaries 2.5(3), 4.4, and 4.8, since in this case $\bar{\phi}=\phi^{-1}$, $\psi\in E$, and hence $(F(\psi),E(\psi))=(F,E)$. When $\phi^{-1}\neq\phi\in E$, part (c) is immediate from Corollaries 2.5(1) and 4.4 because in this case $\bar{\psi} \neq \psi \in F$ and hence $(F(\psi), E(\psi)) = (F, F)$ (and in any case S is (F, F)-rejunctive). When $\phi = -1 \neq 1$, part (c) is immediate from Theorem 3.3(a), since in this case $(F(\psi), E(\psi)) = (F, E)$. Finally, (d) is essentially covered by (ii)(b) and Theorem 3.3(b).

COROLLARY 5.12. *When E is finite in Theorem* 5.8, *then $\phi \in F$ and S is conjoint, and S is rejunctive iff* $1 - \det A = 0 \neq \det(I - A)$.

COROLLARY 5.13. *When E is real-closed in Theorem* 5.8, *then $\phi \in F$, and S is rejunctive iff S is injoint iff S is conjoint iff* $\det(S - S^*)$ *is positive.*

REMARK 5.14. If S is (F, E)-conjoint in Lemma 5.11, then $\det(S - S^*)$ is a nonzero (K, L)-norm in E, but in general not every such norm in E can be obtained in this way, since $\det(S - S^*)$ must also be the sum of two (F, E)-norms here by Theorem 5.6(ii)(b) and its proof.

REMARK 5.15. The following example in the usual complex case (with $i^2 = -1$ as usual) shows that in Theorem 2.4(3) and Corollary 2.5(3) the hypothesis "S is conjoint" cannot in general be weakened to "A is similar to A^{-1}": $S = \operatorname{diag}(1+i, -1+i)$. Here $A = \operatorname{diag}(i, -i)$ is obviously similar to A^{-1} and $q(t) = (t-i)(t+i)$, etc., but by Corollary 5.13 S is not conjoint: $\det(S - S^*) = -4$.

REMARK 5.16. Fact 5.4 gives us, when -1 is a norm, a trivial 1×1 example which is conjoint but not injoint. However, Lemma 5.11 makes it fairly easy to construct a nontrivial 2×2 example of the same phenomenon, even for pairs (F, E) which are "imbedded" in the usual complex case. (By Theorem 5.1 no such examples exist in the latter case itself, of course.) For example, let E be the rational field, $i^2 = -1$ as usual, $F = E(i)$, and

$$S = \begin{bmatrix} 1+i & 1+i \\ -1+i & 1-i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + i \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}.$$

The easiest way to see that S is conjoint here is to verify that

$$C^*SC = S^* \qquad \text{if} \quad C = \begin{bmatrix} 0 & i \\ 1 & 0 \end{bmatrix},$$

and the easiest way to see that S is not injoint is to use Theorem 5.8(ii)(a), (b), since $\det(S - S^*) = 12$ is not an (F, E)-norm. However, the conjointness can also be verified by Lemma 5.11, since $L=E(\sqrt{-3})$, $K=E(i,\sqrt{-3})$, so 12 is the (K,L)-norm of $(3-\sqrt{-3})+i(3+\sqrt{-3})$. Also the non-injointness can be verified directly from elementary properties of rational numbers, using the second form for S above, as $S=I+iH$ with $H^*=H$.

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